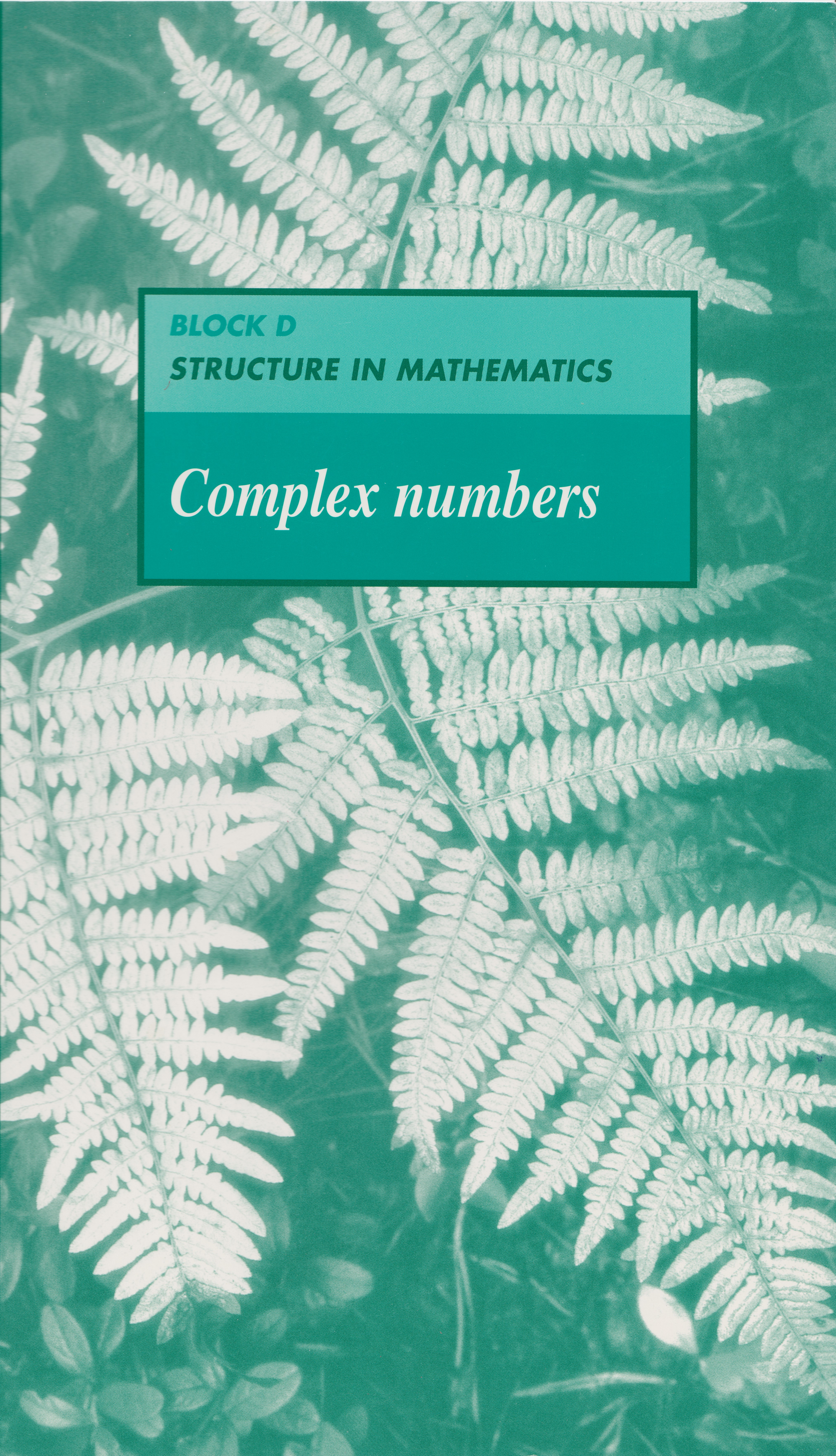
MS221 Chapter D1

The Open University

A second level interdisciplinary course

Exploring Mathematics



BLOCK D

STRUCTURE IN MATHEMATICS

Complex numbers

CHAPTER

D1

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Complex numbers

Prepared by the course team

About this course

This course, MS221 *Exploring Mathematics*, and the courses MU120 *Open Mathematics* and MST121 *Using Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MS221 uses the software program Mathcad (MathSoft, Inc.) to investigate mathematical concepts and as a tool in problem solving. This software is provided as part of the course.

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Contents

Introduction to Block D

Block D consists of four chapters:

Chapter D1, *Complex numbers*;

Chapter D2, *Number theory*;

Chapter D3, *Groups*;

Chapter D4, *Proof and reasoning.*

Although each of Chapters D1–D3 introduces a new mathematical topic that has applications in mathematical methods and models, the main focus in this block is a 'pure mathematical' one. The block covers material that provides an introduction to further pure mathematics courses.

An important aspect of pure mathematics is the invention of mathematical structures. Complex numbers, the topic of Chapter D1, form one such structure. Within the real numbers, certain algebraic equations, such as $x^2 + 1 = 0$, have no solutions. If we supplement $\mathbb{R}$ by introducing numbers of the form $a + bi$, where a and b are real and i represents $\sqrt{-1}$, then we obtain a system within which *all* polynomial equations have solutions.

In Chapter D2, we consider numbers of the simplest type, integers. Over the years, these have provided many intriguing conjectures, the study of which has contributed to the development of a wide range of mathematical ideas.

In Chapter D3, we look at another important mathematical structure: the group. Addition and multiplication of real numbers are examples of *operations*. A set with an operation satisfying certain specific properties is called a *group*. These properties, which provide a generalisation of the properties that apply to addition or multiplication of real numbers, are called the *axioms* for a group. The axiomatic approach is of broad relevance in the study of mathematical structures, and is one of the major innovations of mathematics in the last century or so. The study of groups is a large subject area, of wide application both within and outside mathematics.

Chapter D4 concerns 'proof', a topic that has been a theme throughout MS221. Proving that mathematical statements are true, or false, is another important aspect of pure mathematics.

Study guide

Overall, this chapter should take about one week to study. There is a video associated with Section 4, and there is computing work associated with Section 6. The other sections involve text only. Note that Subsection 5.2 will not be assessed.

Sections 1 to 5 should be studied in the order in which they appear. There are two alternative (and equally suitable) study patterns to cover Section 6, depending on how you prefer to manage your computing work.

Study pattern 1

Study session 1: Sections 1 and 2.
Study session 2: Section 3.
Study session 3: Section 4.
Study session 4: Section 5.
Study session 5: Section 6.

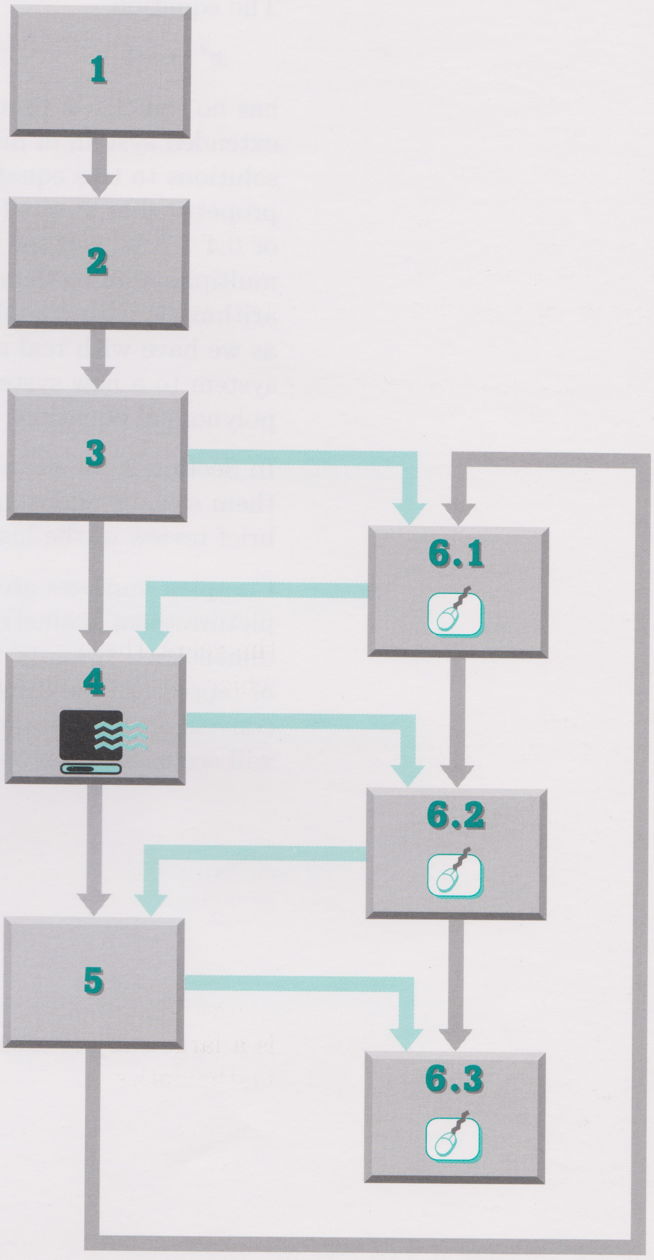
Study pattern 2

Study session 1: Sections 1 and 2.
Study session 2: Section 3 and Subsection 6.1.
Study session 3: Section 4 and Subsection 6.2.
Study session 4: Section 5 and Subsection 6.3.

If you follow the second study pattern, you should do the computing work (in Subsections 6.1, 6.2 and 6.3) *after* you have studied the corresponding text section (Sections 3, 4 and 5, respectively).

There is a video band associated with Section 4, and the most appropriate point to watch it is indicated in the text. However, you could read all the text for Section 4 before watching the video band, if you wish.

The optional Video Band D(i), *Algebra workout – Complex numbers*, could be viewed at any stage during your study of this chapter.



Introduction

The equation

$$x^2 = -1$$

has no solution x that is a real number. In this chapter, we introduce an extended system of numbers, called *complex* numbers, which does include solutions to this equation. We introduce a number, denoted by i, with the property that $i^2 = -1$. More generally, we consider numbers such as $2 + 3i$ or $0.4 - 2.8i$, and see how we can extend operations such as addition and multiplication to them. We do this in a way that allows us to perform arithmetic with complex numbers using the same types of manipulations as we have with real numbers. In this way we extend our real number system to a new system with remarkable properties. For instance, all polynomial equations have solutions in this new system!

In Section 2, we define complex numbers, and we define operations on them such as addition and multiplication. Before that, we start with a brief review of the history of complex numbers.

Complex numbers are introduced for purely algebraic reasons, but we can picture them geometrically, by extending the number line to two dimensions. It turns out that the geometric view leads to alternative ways of representing a complex number. These alternative representations are convenient for certain types of arithmetic with complex numbers, as you will see in Sections 3–5.

1 The origins of complex numbers

The origins of the concept of numbers are lost in prehistory. Recent research suggests that numbers for counting developed before writing, and that writing actually arose out of a need to communicate information about numbers. As well as natural numbers, fractions were used by the Mesopotamians of the second millennium BC (in what is now Iraq) and by the ancient Egyptians. The Rhind papyrus, now in the British Museum, dates from about 1600 BC, and shows a range of techniques for handling fractions.

Numbers arising from geometry were a matter of early interest. Many ancient cultures gave the approximate value 3 for the ratio of the circumference to the diameter of a circle, which we denote by π. The great Sicilian mathematician Archimedes (c. 287–212 BC) proved that π lies between 223/71 and 22/7, and the remarkably accurate value 355/113 was obtained by the Chinese mathematician Tsu Ch'ung-chih (430–501 AD).

We now know that π is not exactly equal to any rational number. Nor is $\sqrt{2}$, which represents the length of the diagonal of a square with sides of length 1. When Greek mathematicians came to realise that $\sqrt{2}$ is not rational, it was a matter of some concern, for their existing idea of 'number' was confined to rationals; so there was some resistance to the acceptance of $\sqrt{2}$ as a number.

It was not until the nineteenth century that mathematicians such as Georg Cantor (1845–1918) and Richard Dedekind (1831–1916) provided a sound foundation to the theory of real numbers, encompassing irrational numbers such as $\sqrt{2}$, and also π and e, which were by then recognised as yet more awkward, for they are not even solutions to any polynomial equation.

Negative numbers were also controversial. Italian mathematicians of the Renaissance derived a method for solving cubic equations, and they found that the method sometimes arrived at an answer that was not within their existing number system. To us, this 'problem' answer is simply a negative number!

Another technique, which Italians of the sixteenth century were exploring for solving algebraic problems, came up with numbers like 'the square root of minus fifteen'. This is where our present story really starts.

Many extensions to the idea of number can be associated with a wish to be able to solve certain types of equation. To solve the equations

$$x + 5 = 0 \quad \text{and} \quad x^2 = 2,$$

for example, we need, for the first, negative numbers, and, for the second, irrational numbers. In this chapter we look at the solution of equations such as

$$x^2 + 1 = 0.$$

To do this we need to introduce yet more 'numbers'. Mathematicians found these new numbers easy to use, but were concerned for a long time about their 'reality'. The new numbers were finally put on a firm footing in the nineteenth century – but let us start the story at its beginning.

Quadratic equations

In trying to find a rectangle with perimeter 20 and area 40, Gerolamo Cardano (1501–1576) introduced two variables – one for the width and one for the height. If we call these variables x and y, then the problem requires us to solve the simultaneous equations

$$x + y = 10 \quad \text{and} \quad xy = 40. \tag{1.1}$$

Eliminating the variable y, this reduces to solving the quadratic equation

$$x^2 - 10x + 40 = 0.$$

Now a quadratic equation

$$ax^2 + bx + c = 0$$

can be solved using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In Cardano's example, with $a = 1$, $b = -10$ and $c = 40$, the formula gives one solution as $x = 5 + \sqrt{-15}$, and it follows that $y = 10 - x = 5 - \sqrt{-15}$. Even though these solutions involve $\sqrt{-15}$, the square root of a negative number, Cardano found that he could treat them like any real number.

To show that he had solved equations (1.1), he argued as follows. Substituting in the first equation gives

$$x + y = (5 + \sqrt{-15}) + (5 - \sqrt{-15}) = 10;$$

substituting in the second equation gives

$$xy = (5 + \sqrt{-15})(5 - \sqrt{-15}) = 5^2 - (\sqrt{-15})^2 = 25 - (-15) = 40;$$

and both are as required. In doing this, all he had to assume was that squaring undoes the square root operation regardless of the value of the number involved; that is, that $(\sqrt{-15})^2 = -15$.

Although this in some sense 'solved' his original problem, not even Cardano felt comfortable about his use of $\sqrt{-15}$. Indeed, he suggested treating expressions involving it as if they were ordinary numbers, and 'leaving aside the mental tortures involved'.

Later mathematicians continued to treat the occurrence of such 'numbers' with suspicion. In 1637 René Descartes (1596–1650) still felt that they were 'imaginary numbers, for clearly they are creatures of the imagination'. Isaac Newton (1642–1727), who perhaps had a firmer understanding of modelling, interpreted their appearance as signifying that 'the problem cannot be solved'. This is clearly the appropriate interpretation in the context of Cardano's original problem of finding a rectangle of perimeter 20 and area 40, since no such rectangle exists.

Activity 1.1 Squaring square roots

Evaluate the product $(3 + \sqrt{-2})(3 - \sqrt{-2})$.

Comment

We find $(3 + \sqrt{-2})(3 - \sqrt{-2}) = 3^2 - (\sqrt{-2})^2 = 9 - (-2) = 11$.

Cubic equations

These controversial square roots of negative numbers took on further significance when Rafael Bombelli (1526–1573) used them to find real solutions to certain problems.

Suppose this time we have the simultaneous equations

There is no need to follow this discussion in detail.

$$x^2 + y = 21 \quad \text{and} \quad xy = 20.$$

By eliminating y, we obtain

$$x^3 - 21x + 20 = 0. \tag{1.2}$$

Bombelli knew that a cubic equation of the form

$$x^3 + cx + d = 0$$

can be solved using the formula

This method of solution is due to Cardano.

$$x = g^{1/3} - \tfrac{1}{3}cg^{-1/3}, \tag{1.3}$$

where

$$g = -\frac{d}{2} + \frac{\sqrt{3(4c^3 + 27d^2)}}{18}.$$

The cubic equation (1.2) has $c = -21$ and $d = 20$, from which we can calculate that

$$g = -10 + 9\sqrt{-3},$$

an expression involving the square root of a negative number.

To use equation (1.3), we need to find a cube root of g.

Activity 1.2 Checking a cube root

(a) Show that

$$(2 + \sqrt{-3})^3 = -10 + 9\sqrt{-3}.$$

(b) Show that

$$(2 + \sqrt{-3}) \times \frac{2 - \sqrt{-3}}{7} = 1.$$

Comment

(a) We evaluate the cube using the Binomial Theorem:

See Chapter B1, Section 5; the expansion of $(x + y)^3$ is $x^3 + 3x^2y + 3xy^2 + y^3$.

$$\begin{aligned}(2 + \sqrt{-3})^3 &= 2^3 + 3 \times 2^2(\sqrt{-3}) + 3 \times 2(\sqrt{-3})^2 + (\sqrt{-3})^3 \\ &= 8 + 12\sqrt{-3} + 6(-3) + (-3)\sqrt{-3} \\ &= -10 + 9\sqrt{-3}.\end{aligned}$$

Hence $2 + \sqrt{-3}$ *is* a cube root of $-10 + 9\sqrt{-3}$.

(b) We have

$$(2 + \sqrt{-3}) \times \frac{2 - \sqrt{-3}}{7} = \frac{2^2 - (\sqrt{-3})^2}{7} = \frac{7}{7} = 1.$$

The result of Activity 1.2(a) shows that $2+\sqrt{-3}$ is a cube root of $g=-10+9\sqrt{-3}$. If we take $g^{1/3}=2+\sqrt{-3}$, then the result of Activity 1.2(b) shows that

$$g^{-1/3}=\frac{1}{g^{1/3}}=\frac{1}{2+\sqrt{-3}}=\frac{2-\sqrt{-3}}{7}.$$

These expressions still involve the square root of a negative number. But using equation (1.3), we obtain

$$\begin{aligned} x=g^{1/3}-\tfrac{1}{3}cg^{-1/3}&=(2+\sqrt{-3})-\tfrac{1}{3}(-21)\left(\frac{2-\sqrt{-3}}{7}\right)\\ &=2+\sqrt{-3}+2-\sqrt{-3}=4. \end{aligned}$$

Remarkably, all the square roots of negative numbers have cancelled out to give a real number, $x=4$. You can easily check that $x=4$ does satisfy equation (1.2). Returning to the original pair of equations, we use $xy=20$ to give $y=5$, and we observe that together $x=4$ and $y=5$ do give a solution to $x^2+y=21$.

This example shows that in some circumstances it may be *useful* to work with square roots of negative numbers!

Summary of Section 1

In this section, you saw a little of the early history of the use of square roots of negative numbers. You saw how such square roots arise in the solution of quadratic and cubic equations, and how to manipulate some arithmetic expressions involving these square roots.

2 Complex numbers

In this section we shall develop an arithmetic of numbers such as $5 + \sqrt{-15}$, involving the square roots of negative numbers.

We begin by introducing some notation. For any positive real number r we can write

$$\sqrt{-r} = \sqrt{r(-1)} = \sqrt{r}\sqrt{-1},$$

where $\sqrt{r}$ is taken to mean the positive square root. Since $r > 0$, $\sqrt{r}$ is just a real number, so we can express the square roots of all negative numbers as a real multiple of the square root of -1. Leonhard Euler (1707–1783) introduced the symbol i for $\sqrt{-1}$, and using this we have $\sqrt{-r} = (\sqrt{r})i$, which can be written more succinctly as $\sqrt{r}\,i$.

Using this notation we can, for example, write $5 + \sqrt{-15}$ as $5 + \sqrt{15}\,i$. In fact, we can represent all the new numbers with which we shall be concerned by using i.

> **Complex numbers**
>
> A **complex number** is of the form $a + bi$, where a and b may be any real numbers and where $i^2 = -1$.
>
> We use $\mathbb{C}$ to represent the set of all complex numbers.
>
> For a complex number $z = a + bi$, we call a the **real part** of z and b the **imaginary part** of z. We write $\mathrm{Re}(z)$ for the real part, a, and $\mathrm{Im}(z)$ for the imaginary part, b.

Notice that the 'imaginary part' of a complex number is a real number. For example, the imaginary part of the complex number $3 - 5i$ is -5, *not* $-5i$.

Examples of complex numbers are $3 + 4i$, $12 - 5i$ and $-1 - 2i$. However, there is no restriction on the real numbers that can be used as the real and the imaginary parts. They can be rationals like $\frac{3}{2}$, roots like $\sqrt{2}$, numbers like π, or any combination of such real numbers.

In particular $0 + 2i$, $-3 + 0i$ and $0 + 0i$ are all complex numbers, and for simplicity we write these numbers as $2i$, -3 and 0 respectively, omitting 0s where possible. Also, we usually write numbers like $3 + 1i$ and $-1 - 1i$ as $3 + i$ and $-1 - i$ respectively, suppressing the redundant symbol 1.

This notational device demonstrates that the real numbers are included in the set of complex numbers. For example, the real number 5 is identified with the complex number $5 + 0i$.

Activity 2.1 Real and imaginary parts

(a) What are the real and imaginary parts of each of: $2i$; -3?

(b) Solve the quadratic equation

$$z^2 - 6z + 25 = 0,$$

giving the solutions as complex numbers z_1 and z_2. Write down the real and imaginary parts of each of these numbers.

Solutions are given on page 51.

We sometimes write the imaginary part of a complex number *after* the i. For example, we may write $1 + i\sqrt{2}$ in preference to $1 + \sqrt{2}\,i$. This is often done to avoid any possible confusion between $\sqrt{2}\,i$ and $\sqrt{2i}$.

We shall discuss the meaning of $\sqrt{2i}$ later in this chapter.

2.1 The arithmetic of complex numbers

So far, we have performed arithmetic with complex numbers as if they were real numbers. This turns out to be justified, but it would be incautious simply to assume that it is. To be confident that this is correct, we should be systematic.

First we must define just what we mean by addition and multiplication of complex numbers. Then, using our definitions, we should verify that these arithmetic operations on complex numbers do share various properties with the corresponding operations on real numbers, which form the basis of ordinary arithmetic.

Examples of the sort of properties that we are referring to are

$$(a + b) + c = a + (b + c),$$

which states that, in adding three real numbers, the order in which the two additions are performed does not matter, and

$$a \times (b + c) = a \times b + a \times c,$$

which we use when expanding brackets.

'New' operations are not always so obliging as to satisfy such properties. For example, multiplication of matrices does *not* satisfy the rule $\mathbf{AB} = \mathbf{BA}$, although the equivalent rule for real numbers does hold. Fortunately, operations on complex numbers do not have hidden catches, and turn out to behave just like their equivalents for real numbers.

See MST121 Chapter B2.

First, we shall say what we mean by equality of complex numbers. It would be easy to overlook this basic idea, but a clear definition of equality is necessary to a sound development.

Complex numbers are built up from their real and imaginary parts, so it seems reasonable that two complex numbers should be equal if their real parts are equal and their imaginary parts are equal.

Equality of complex numbers

Let z and w be two complex numbers. Then

$$z = w$$

means that

$$\mathrm{Re}(z) = \mathrm{Re}(w) \quad \text{and} \quad \mathrm{Im}(z) = \mathrm{Im}(w).$$

We next look at addition and multiplication of complex numbers. For now, we shall continue to assume that arithmetic for complex numbers is just like that for reals. Later, having defined the operations, we shall look at what is needed to justify this assumption.

Here is how we add the two complex numbers $3 + 4i$ and $12 + 5i$:

$$\begin{aligned}(3 + 4i) + (12 + 5i) &= (3 + 12) + (4i + 5i)\\ &= 15 + (4 + 5)i\\ &= 15 + 9i.\end{aligned}$$

All we have done is to add the real parts and to add the imaginary parts.

You may notice a similarity to the addition of vectors, described in Chapter B2, Section 1, where we add the components separately.

Addition of complex numbers

Let $z = a + bi$ and $w = c + di$. Then

$$z + w = (a + c) + (b + d)i.$$

It is not necessary to remember this formal definition. Addition can be performed from first principles.

Activity 2.2 Addition of complex numbers

Add the two complex numbers $-3 + 4i$ and $12 - 5i$.

Comment

The sum is

$$\begin{aligned}(-3 + 4i) + (12 - 5i) &= (-3 + 12) + (4 - 5)i \\ &= 9 - i.\end{aligned}$$

Multiplying two complex numbers is only slightly harder. We expand the expression for the product in the usual way, and simplify using $i^2 = -1$. For example:

$$\begin{aligned}(3 + 4i) \times (12 + 5i) &= 3 \times 12 + 3 \times 5i + 12 \times 4i + 4 \times 5i^2 \\ &= 36 + 15i + 48i - 20 \\ &= 16 + 63i.\end{aligned}$$

This illustrates the method of multiplying general complex numbers.

Multiplication of complex numbers

Let $z = a + bi$ and $w = c + di$. Then

$$z \times w = (ac - bd) + (ad + bc)i.$$

Again there is no need to remember this formula. Like addition, multiplication of complex numbers can be performed from first principles. We need only remember that $i^2 = -1$.

Activity 2.3 Multiplication of complex numbers

(a) Find the product

$$(-3 + 2i) \times (6 - 5i).$$

(b) Expand the general product

$$(a + bi) \times (c + di)$$

and verify that you obtain the expression in the definition of multiplication of complex numbers given in the box above.

Solutions are given on page 51.

2.2 Properties of addition and multiplication

We now turn to properties of addition and multiplication that are used when doing ordinary arithmetic. For any real numbers a, b and c, we have the following properties.

$$a + b = b + a \tag{2.1}$$

$$(a + b) + c = a + (b + c) \tag{2.2}$$

$$a \times b = b \times a \tag{2.3}$$

$$(a \times b) \times c = a \times (b \times c) \tag{2.4}$$

$$a \times (b + c) = a \times b + a \times c \tag{2.5}$$

Operations in general

Property (2.1) says that, when adding two real numbers, it does not matter which comes first. (We obtain the same result from $5 + 2$ and from $2 + 5$, for example.) Property (2.3) makes a similar statement about multiplication: the order of the two numbers being multiplied does not matter. (For example, $(-2) \times 4$ gives the same result as $4 \times (-2)$.) For any operation, we say that it is *commutative* if the order in which elements are combined does not matter. Formally, an operation $*$ on a set X is **commutative** if, for all a and b in X, we have

$$a * b = b * a.$$

So equation (2.1) above states that addition of real numbers is commutative, and equation (2.3) states that multiplication of real numbers is commutative. Not all operations are commutative, however. For example, as noted earlier, multiplication of matrices is *not* commutative.

Equation (2.2) states that when we add three numbers, it does not matter which two are added first. Equation (2.4) has a similar structure, but applied to multiplication: when three real numbers are multiplied, it does not matter which two are multiplied first. For an operation $*$ on a set X, we say that $*$ is **associative** if, for all a, b and c in X, we have

$$(a * b) * c = a * (b * c).$$

Equation (2.2) states that addition of real numbers is associative, and equation (2.4) states that multiplication of real numbers is associative.

Finally, we express the property in equation (2.5) by saying that, for real numbers, multiplication is **distributive** over addition. This again is an idea that could be generalised as a possible property of any two operations on the same set.

Activity 2.4 Commutative or associative?

For each of the operations in parts (a)–(c), say whether or not it is: (i) commutative; (ii) associative.

(a) Subtraction of real numbers.

(b) Addition of 2×2 matrices.

(c) Multiplication of 2×2 matrices.

Comment

(a) Subtraction of real numbers is *not* commutative. For example,

$$2 - 3 \neq 3 - 2.$$

Subtraction is not associative either. For example, we obtain different values for

$$(2-3) - 5 = -6 \quad \text{and} \quad 2 - (3-5) = 4.$$

(b) Addition of 2×2 matrices *is* commutative. For any two 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$, we do have that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

Properties of matrix operations were discussed in MST121 Chapter B2.

Addition of 2×2 matrices is also associative. For any 2×2 matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, we do have that $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

(c) As remarked earlier, multiplication of 2×2 matrices is *not* commutative. We can find 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ for which $\mathbf{AB} \neq \mathbf{BA}$.

Multiplication of 2×2 matrices *is* associative, though. For any 2×2 matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, we do have that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

All of the properties (2.1)–(2.5) extend to complex numbers, as we would hope. That is, if z, w and u are any three complex numbers, the following properties hold.

$$z + w = w + z \tag{2.6}$$

$$(z + w) + u = z + (w + u) \tag{2.7}$$

$$z \times w = w \times z \tag{2.8}$$

$$(z \times w) \times u = z \times (w \times u) \tag{2.9}$$

$$z \times (w + u) = z \times w + z \times u \tag{2.10}$$

We can verify that addition of complex numbers is commutative (equation (2.6)) as follows. Consider any two complex numbers $z = a + bi$ and $w = c + di$. Then, by the definition of addition of complex numbers,

$$z + w = (a + c) + (b + d)i$$

and similarly

$$w + z = (c + a) + (d + b)i.$$

But, since a, b, c and d are real numbers, we know that

$$a + c = c + a \quad \text{and} \quad b + d = d + b.$$

That is, the real parts of the complex numbers $z + w$ and $w + z$ are equal, and so are their imaginary parts. Hence these complex numbers are equal:

$$z + w = w + z.$$

Activity 2.5 Multiplication is commutative

By considering complex numbers $z = a + bi$ and $w = c + di$, show that multiplication of complex numbers is commutative (equation (2.8)).

A solution is given on page 51.

The other properties (2.7), (2.9) and (2.10) can be verified in a similar way. The algebraic details of this are rather messy, and are omitted.

The real numbers 0 and 1 play a special role in the following properties of real numbers.

$$a + 0 = a \tag{2.11}$$

$$0 \times a = 0 \tag{2.12}$$

$$1 \times a = a \tag{2.13}$$

The corresponding complex numbers, $0 = 0 + 0i$ and $1 = 1 + 0i$, play a similar role. For any complex number z, we have the following.

$$z + 0 = z \tag{2.14}$$

$$0 \times z = 0 \tag{2.15}$$

$$1 \times z = z \tag{2.16}$$

Activity 2.6 The numbers 0 and 1

Using the definitions of addition and multiplication for complex numbers, verify each of equations (2.14)–(2.16).

A solution is given on page 51.

2.3 Subtraction and division

We now turn to subtraction and division of complex numbers. It would seem natural that, say,

$$(4 + 6i) - (5 + 2i) = (4 - 5) + (6 - 2)i = -1 + 4i.$$

It is less obvious what

$$\frac{4 + 6i}{5 + 2i}$$

should be. To deal with division, z/w, of complex numbers z and w, we shall proceed in two steps. We first define the *reciprocal*, $1/w$, of a complex number w, and then define z/w as $z \times 1/w$.

But before looking at division, we deal with subtraction. A similar approach is used to define $z - w$. First define the *negative*, $-w$, of a complex number w, and then define $z - w$ as $z + (-w)$.

For each real number a, there is a real number $-a$ that satisfies

$$a + (-a) = 0. \tag{2.17}$$

We want this property to hold for complex numbers also. We can ensure this by defining the negative of a complex number as follows.

Negative of a complex number

Let $z = a + bi$. Then the **negative** of z is

$$-z = -a - bi.$$

You can readily check that, with this definition of $-z$, we have

$$z + (-z) = 0. \tag{2.18}$$

For real numbers r and s, we know that $r - s$ is the same as $r + (-s)$. So for complex numbers z and w, it is reasonable to define $z - w$ as $z + (-w)$. If $z = a + bi$ and $w = c + di$, then

$$z + (-w) = (a + bi) + (-c - di) = (a - c) + (b - d)i.$$

Thus $z - w$ is obtained by simply subtracting each of the real and imaginary parts of w from those of z.

Subtraction of complex numbers

Let $z = a + bi$ and $w = c + di$. Then

$$z - w = (a - c) + (b - d)i.$$

Here is a subtraction for you to try.

Activity 2.7 A subtraction

If $z = -3 + 4i$ and $w = 12 - 5i$, find $z - w$.

A solution is given on page 51.

With this definition, we can perform the usual algebraic manipulations involving subtraction. For instance, if z, w and u are three complex numbers and $z = u + w$, then it follows that $z - w = u$.

We now look at division. For each real number $a \neq 0$ there is a number a^{-1}, called the reciprocal of a, that satisfies

$$a \times a^{-1} = 1. \tag{2.19}$$

If s and r are real numbers with $r \neq 0$, then we have $s/r = s \times r^{-1}$. So, for a complex number z with $z \neq 0$, we want the reciprocal z^{-1} to be the complex number for which

$$z \times z^{-1} = 1. \tag{2.20}$$

Then we can define w/z to be $w \times z^{-1}$.

For example, consider $z = 3 + 4i$. Suppose this has reciprocal $z^{-1} = x + yi$. Then from equation (2.20) we know that $1 = z \times z^{-1} = (3 + 4i) \times (x + yi)$. Hence

$$1 = (3x - 4y) + (3y + 4x)i.$$

For equality, we must have both the real and imaginary parts equal, and so the above equation for complex numbers gives the following two equations:

$$3x - 4y = 1,$$
$$4x + 3y = 0.$$

These equations have solutions $x = \frac{3}{25}$ and $y = -\frac{4}{25}$. Hence $z^{-1} = \frac{3}{25} - \frac{4}{25}i$. Thus the reciprocal of the complex number $z = 3 + 4i$ can be written as

$$z^{-1} = \frac{3 - 4i}{25}.$$

There is a pattern here. The top of the expression for z^{-1} is obtained from z by changing the sign of the imaginary part, while the bottom is the sum of the squares of both the real and imaginary parts, since $25 = 3^2 + 4^2$. Does this pattern generalise?

For any complex number $a + bi$ and any real number $r \neq 0$, the notation

$$\frac{a + bi}{r}$$

means

$$\frac{a}{r} + \frac{b}{r}i.$$

Activity 2.8 Checking the pattern

Let $z = a + bi$ and $w = \dfrac{a - bi}{a^2 + b^2}$. Show that $z \times w = 1$.

Comment

We can write the product $z \times w$ as

$$(a + bi) \times \left(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i \right)$$

and use the formula for the multiplication of complex numbers. Thus

$$z \times w = \left(\frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} \right) + \left(\frac{-ab}{a^2 + b^2} + \frac{ab}{a^2 + b^2} \right) i = 1 + 0i = 1,$$

as required.

The result of Activity 2.8 justifies the following definition.

Reciprocal of a complex number

Let $z = a + bi$ $(z \neq 0)$. Then the **reciprocal** of z is

$$z^{-1} = \frac{a - bi}{a^2 + b^2}.$$

Now that we know how to calculate reciprocals, we can perform division.

Division of complex numbers

Let w and $z \neq 0$ be two complex numbers. Then

$$\frac{w}{z} = w \times z^{-1}.$$

Suppose that $u \times z = w$, where z, u and w are complex. As long as $z \neq 0$, we can rearrange this equation just as we can with real numbers, to obtain $u = w/z$.

Activity 2.9 Division by a real number

Suppose that r is a non-zero real number.

(a) If r is regarded as a complex number $r + 0i$, what does the definition of the reciprocal of a complex number give for r^{-1}?

(b) If $z = a + bi$ is an arbitrary complex number, show that the definition of division of complex numbers gives

$$\frac{z}{r} = \frac{a}{r} + \frac{b}{r} i.$$

Comment

(a) Applying the definition of a complex reciprocal to $r + 0i$, we obtain

$$r^{-1} = \frac{r - 0i}{r^2 + 0^2} = \frac{1}{r} + 0i.$$

This is as expected!

(b) The definition of complex division gives

$$\begin{aligned}\frac{z}{r} &= (a + bi) \times r^{-1} \\ &= (a + bi) \times \left(\frac{1}{r} + 0i\right) \\ &= \frac{a}{r} + \frac{b}{r}i,\end{aligned}$$

as required. (This justifies our earlier use of the notation $\frac{a+bi}{r}$.)

Activity 2.10 Division by a complex number

Let $w = -3 + 4i$ and $z = 12 - 5i$. Find z^{-1} and hence solve the equation $u \times z = w$.

Comment

From the definition,

$$z^{-1} = \frac{12 + 5i}{12^2 + 5^2} = \frac{12 + 5i}{169}.$$

If $u \times z = w$ then

$$u = \frac{w}{z} = w \times z^{-1} = (-3 + 4i) \times \frac{12 + 5i}{169} = \frac{-56 + 33i}{169} = \frac{-56}{169} + \frac{33}{169}i.$$

Though the formula for the reciprocal is not complicated, it is not essential to remember it. To find the reciprocal of a complex number $z = a + bi$, you can write $z^{-1} = \frac{1}{a + bi}$ and then multiply both top and bottom by $a - bi$; that is,

$$z^{-1} = \frac{1}{a + bi} = \frac{a - bi}{(a + bi) \times (a - bi)} = \frac{a - bi}{a^2 + b^2}.$$

We make the following definition.

Complex conjugate of a complex number

Let $z = a + bi$. Then then the **complex conjugate** of z is

$$\overline{z} = a - bi.$$

Notice that the complex conjugate of $\overline{z}$ is just z again.

Complex conjugates appear in a variety of contexts. In particular they can be used to simplify any complex 'fraction' (that is, make its denominator real) by multiplying top and bottom by the complex conjugate of the denominator.

Activity 2.11 Complex conjugates

(a) Let $z = a + bi$. Calculate $z \times \overline{z}$.

(b) Simplify the complex fraction

$$\frac{1+2i}{5-i}.$$

(c) Show that, if a, b and c are real numbers (with $a \neq 0$), then the solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

are either real or a pair of complex conjugates.

Comment

(a) $z \times \overline{z} = (a + bi) \times (a - bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2$.

(b) Multiply top and bottom by the complex conjugate of $5 - i$:

$$\begin{aligned}\frac{1+2i}{5-i} &= \frac{(1+2i)\times(5+i)}{(5-i)\times(5+i)} \\ &= \frac{5+i+10i+2i^2}{25-i^2} \\ &= \frac{3+11i}{26}.\end{aligned}$$

(c) The formula for the solution of a quadratic equation gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 \geq 4ac$, then the solutions are real. If $b^2 < 4ac$, then

$$b^2 - 4ac = (-1)(4ac - b^2)$$

where $4ac - b^2 > 0$, and so

$$x = \frac{-b \pm \sqrt{(-1)(4ac - b^2)}}{2a} = \frac{-b \pm \left(\sqrt{4ac - b^2}\right) i}{2a}.$$

These solutions are complex conjugates of each other, as they are of the form $m \pm ni$, where m and n are real.

Summary of Section 2

A complex number is of the form $z = a + bi$, where a and b are real and $i^2 = -1$. The complex conjugate of z, written $\overline{z}$, is $a - bi$. We can perform arithmetic with complex numbers in the same way as we can with reals.

To do this, we defined the operations $+$ and $\times$ for complex numbers. Then we identified rules used in performing ordinary arithmetic on real numbers, and noted the need to verify that complex numbers also satisfy these rules. In doing this, we introduced the terms 'commutative' and 'associative', which may (or may not) apply to any operation. We also defined the negative and the reciprocal of a complex number, and used these to define subtraction and division.

In performing arithmetic with complex numbers, there is no need to remember the formal definitions of the operations; we simply manipulate complex numbers like real numbers. To simplify a fraction involving complex numbers, multiply the top and bottom of the fraction by the complex conjugate of the denominator.

Exercises for Section 2

Exercise 2.1

Let $z = 2 + i$ and $w = 1 - 2i$. Find each of the complex numbers in parts (a)–(g).

(a) $z + w$ (b) $-w$ (c) $z - w$ (d) $-2z$

(e) $z \times w$ (f) w^{-1} (g) $\dfrac{z}{w}$

Exercise 2.2

Let $z = a + bi$ and $w = c + di$ be any two complex numbers. Show that

$$\overline{z} + \overline{w} = \overline{z + w}.$$

3 The geometry of complex numbers

3.1 The Argand diagram

We can visualise real numbers as lying on the number line (Figure 3.1). Is there a similar way of picturing complex numbers?

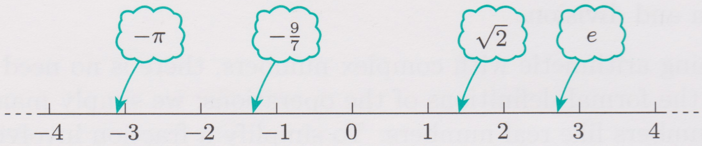


Figure 3.1 The number line

Well, complex numbers are described by two real numbers, the real and imaginary parts, which suggests that we might use points in the plane to represent them. We do this, and represent the complex number $z = x + yi$ by the point (x, y) in the plane. For example, the complex numbers $3 + 4i$, $1 - 2i$ and $-3 + i$ are represented by the points $(3, 4)$, $(1, -2)$ and $(-3, 1)$, respectively (Figure 3.2). Considered as complex numbers, the integers -1, 0, 1, 2 and 3 are represented by the points $(-1, 0)$, $(0, 0)$, $(1, 0)$, $(2, 0)$ and $(3, 0)$, whereas the complex numbers $-i$, i, $2i$ and $3i$ are represented by $(0, -1)$, $(0, 1)$, $(0, 2)$ and $(0, 3)$. Notice how, in Figure 3.2, real numbers are shown on the horizontal axis, whereas points on the vertical axis represent real multiples of i.

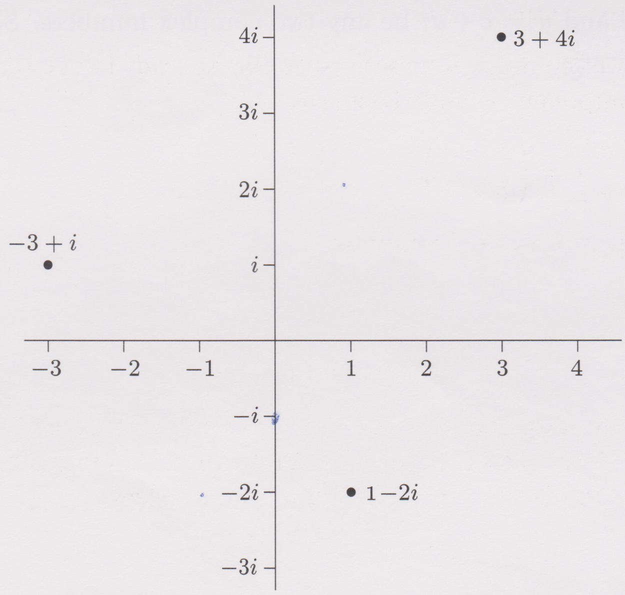


Figure 3.2 Certain complex numbers visualised as points in the plane

Historical note

A representation of complex numbers by points in a plane is often called an **Argand diagram**, after the French mathematician Jean-Robert Argand (1768–1822), whose publication of the idea in 1806 was the first to be generally recognised. The idea had been proposed in 1673 by John Wallis (1616–1703), but for some reason his suggestion was ignored. The idea reappeared in 1797, in a paper written in Danish by Caspar Wessel (1745–1818), but was again overlooked. The idea finally became generally accepted after its exposition by Carl Friedrich Gauss (1777–1855), in 1831.

This representation is also referred to as the **complex plane**.

Activity 3.1 The Argand diagram

Mark the complex numbers $-i$, $1+2i$ and $-1-2i$ on an Argand diagram.

A solution is given on page 51.

Arithmetic and the Argand diagram

How might we visualise addition of complex numbers on an Argand diagram? Suppose that $z = a + bi$ and $w = c + di$. Then, by definition,

$$z + w = (a + c) + (b + d)i.$$

So, for the complex numbers represented by the points (a, b) and (c, d), their sum is represented by the point $(a + c, b + d)$. We just add the coordinates separately.

Thus addition of complex numbers is similar to addition of vectors, in which each component is added separately:

See Chapter B2, Section 1.

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix},$$

where a, b, c and d are real numbers.

Now, the vector

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

can be identified with the point (a, b) in the plane. Since addition of vectors in this form satisfies a parallelogram rule, this rule also holds for complex numbers, as illustrated in Figure 3.3 overleaf.

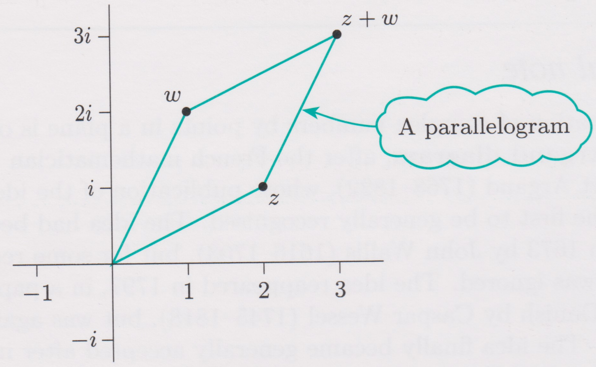


Figure 3.3 The sum, $z + w = 3 + 3i$, of the complex numbers $z = 2 + i$ and $w = 1 + 2i$, shown on an Argand diagram

Activity 3.2 Other operations on Argand diagrams

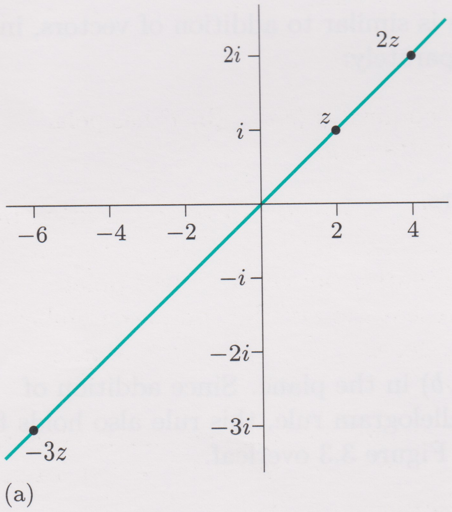
Let $z = 2 + i$ and $w = 1 + 3i$.

(a) Mark z, $2z$ and $-3z$ on an Argand diagram, and describe the geometric relationship between them.

(b) Mark z, w and $z - w$ on an Argand diagram, and describe the geometric relationship between them.

Comment

(a) See Figure 3.4(a). Note that all real multiples of z lie on the straight line passing through z and the origin.



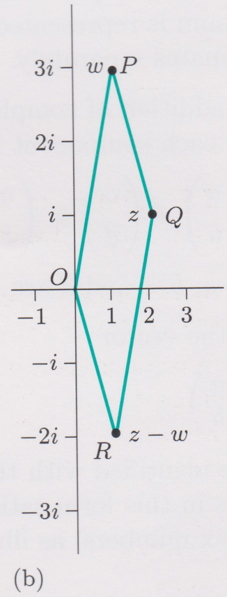


Figure 3.4 Argand diagrams showing: (a) real multiples of a complex number; (b) subtraction of complex numbers

(b) See Figure 3.4(b). We have $z - w = (2 + i) - (1 + 3i) = 1 - 2i$, which is represented by $(1, -2)$ on the Argand diagram. Notice that $OPQR$ is a parallelogram, whose corners are points representing the complex

numbers 0, w, z and $z - w$. We know that $(z - w) + w = z$, and this corresponds to the vector sum

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

which is represented by this parallelogram.

3.2 Polar form

We now look at an alternative representation of complex numbers. This representation is based on polar (rather than Cartesian) coordinates, and it leads to a neat way of expressing multiplication of complex numbers.

First we introduce **polar coordinates** for points in the plane; see Figure 3.5(a). The position of a point P in the plane is represented by $\langle r, \theta \rangle$, where r is the distance OP and θ is the angle between the positive x-axis and the line segment OP. Here $r \geq 0$, and we can choose θ to be in the range $-\pi < \theta \leq \pi$. Other values of θ just give an alternative representation; for example, $\langle 1, 3\pi \rangle$ represents the same point as $\langle 1, \pi \rangle$. We use angle brackets to distinguish polar coordinates from Cartesian coordinates; the latter are given in round brackets, as usual.

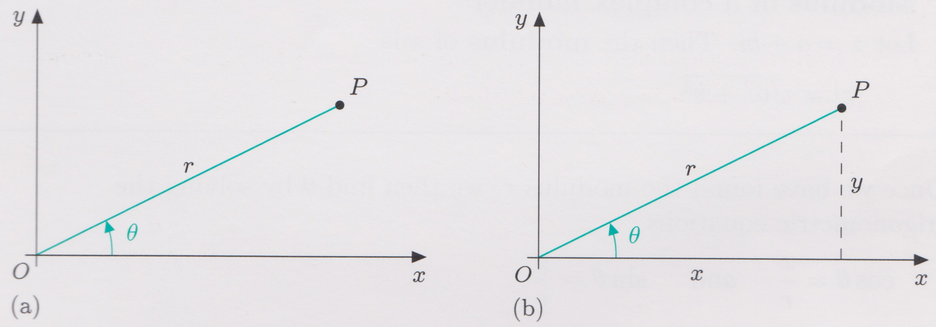


Figure 3.5 (a) A point P with polar coordinates $\langle r, \theta \rangle$ (b) P has polar coordinates $\langle r, \theta \rangle$ and Cartesian coordinates (x, y)

We have seen that a complex number $z = x + yi$ can be represented by the point in the plane with Cartesian coordinates (x, y). This point will also have a representation in polar coordinates, say $\langle r, \theta \rangle$. These polar coordinates give an alternative representation of the complex number z, which we call its **polar form**. We refer to the usual representation, written $x + yi$, as **Cartesian form**.

Whenever there are two representations for a single mathematical idea we need to see how they are related. This is shown in Figure 3.5(b), from which we can read off the relationships

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

These relationships hold in all quadrants.

Hence the complex number with polar form $\langle r, \theta \rangle$ has Cartesian form

$$r\cos\theta + (r\sin\theta)i,$$

which we can also write as

$$r(\cos\theta + i\sin\theta).$$

Activity 3.3 Polar form for complex numbers

Find the Cartesian form for the complex numbers given in polar form as $\langle 1, 0\rangle$, $\langle 1, \pi/4\rangle$ and $\langle 2, \pi/2\rangle$.

Comment

For $\langle 1, 0\rangle$ we have Cartesian form: $1\cos 0 + (1\sin 0)i = 1 + 0i = 1$.

For $\left\langle 1, \frac{\pi}{4}\right\rangle$ we have: $1\cos\left(\frac{\pi}{4}\right) + 1\sin\left(\frac{\pi}{4}\right)i = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$.

For $\left\langle 2, \frac{\pi}{2}\right\rangle$ we have: $2\cos\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{\pi}{2}\right)i = 0 + 2i = 2i$.

Conversely, suppose that we have a complex number in Cartesian form, $z = x + yi$, and we want to find the corresponding polar form. This time we want to find r and θ in terms of x and y. Look again at Figure 3.5(b). By Pythagoras' Theorem we have $r^2 = x^2 + y^2$, and hence

$$r = \sqrt{x^2 + y^2}$$

(where we take the positive square root). We call this distance the **modulus** of the complex number, written $|z|$.

> **Modulus of a complex number**
> Let $z = a + bi$. Then the **modulus** of z is
> $$|z| = \sqrt{a^2 + b^2}.$$

Once we have found the modulus r, we then find θ by solving the trigonometric equations

$$\cos\theta = \frac{x}{r} \quad \text{and} \quad \sin\theta = \frac{y}{r}.$$

We call this angle θ the **argument** of z, and write it as $\arg(z)$.

Example 3.1

Suppose $z = -1 + i$. Find the modulus, argument and polar form of z.

Solution

We have $r^2 = (-1)^2 + 1^2$ and hence the modulus is $r = \sqrt{2}$. To find the argument we need to solve

$$\cos\theta = \frac{-1}{\sqrt{2}} \quad \text{and} \quad \sin\theta = \frac{1}{\sqrt{2}}.$$

To do this we look at an Argand diagram for z (see Figure 3.6) and see that z lies in the second quadrant. Hence we expect a value for θ lying between $\pi/2$ and π.

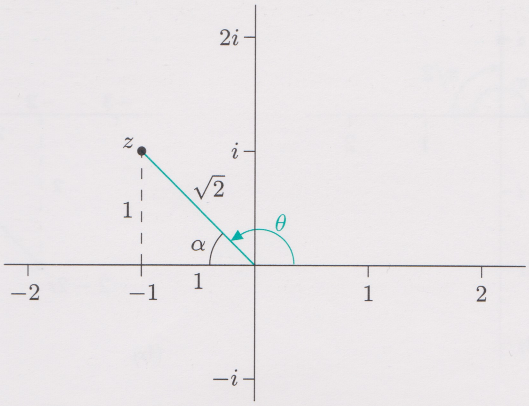


Figure 3.6 An Argand diagram showing $-1+i$

The angle α shown on the diagram lies between 0 and $\pi/2$ and has $\cos\alpha = 1/\sqrt{2}$. Thus $\alpha = \arccos(1/\sqrt{2}) = \pi/4$. Then the argument, θ, of z is

$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Since $r = \sqrt{2}$ and $\theta = 3\pi/4$, the polar form for the complex number $-1+i$ is $\left\langle \sqrt{2}, 3\pi/4 \right\rangle$.

In Example 3.1 we found a particular value for the argument θ. However, there is some ambiguity in this value. We could have ended up pointing in the same direction after completing any additional number of complete rotations around the origin in either direction. In general, the argument of a complex number is not unique – it can differ by any integer multiple of 2π radians.

Ambiguity of polar forms of complex numbers

The polar forms $\langle r, \theta\rangle$ and $\langle s, \phi\rangle$ represent the same complex number if and only if

$$r = s \quad \text{and} \quad \theta - \phi = 2\pi m, \quad \text{where } m \text{ is an integer.}$$

In changing from the Cartesian to the polar form of a complex number, you need to be aware of the choices that are available to you in the latter representation.

Activity 3.4 Polar form

(a) Express in polar form each of the following complex numbers.

(i) i (ii) -2

(b) Express the complex number $z = -2-2i$ in polar form. Give two alternative polar forms.

Comment

(a) These complex numbers are shown on an Argand diagram in Figure 3.7(a) overleaf. Their polar forms are:

(i) $\langle 1, \pi/2\rangle$, (ii) $\langle 2, \pi\rangle$.

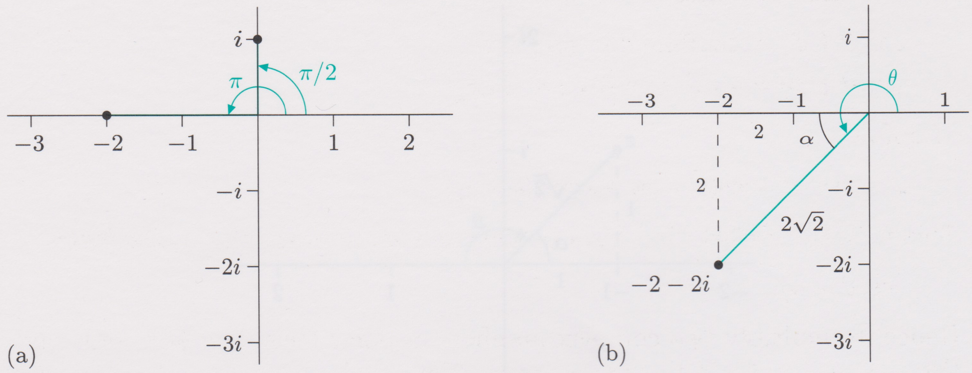


Figure 3.7 Obtaining polar forms for: (a) i, -2; (b) $-2-2i$

(b) Here $r^2 = (-2)^2 + (-2)^2 = 8$ and hence $r = 2\sqrt{2}$. To find the argument, we need to locate which quadrant z lies in. An Argand diagram helps; see Figure 3.7(b). This shows that z lies in the third quadrant. We can also see that the angle α in Figure 3.7(b) has

$$\cos\alpha = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}, \quad \text{so} \quad \alpha = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

The argument θ is measured anticlockwise from the positive x-axis, so $\theta = \pi + \pi/4 = 5\pi/4$. This is one value of the argument, and gives the polar form of $-2-2i$ as $\left\langle 2\sqrt{2}, 5\pi/4\right\rangle$.

Going an extra complete rotation *anticlockwise* around the origin adds 2π to the argument and gives the alternative polar form $\left\langle 2\sqrt{2}, 13\pi/4\right\rangle$. Going one complete rotation *clockwise* around the origin subtracts 2π from the argument and gives the polar form $\left\langle 2\sqrt{2}, -3\pi/4\right\rangle$.

Here $5\pi/4 + 2\pi = 13\pi/4$.

Here $5\pi/4 - 2\pi = -3\pi/4$.

In general, $-2-2i$ is represented by the polar form $\left\langle 2\sqrt{2}, 5\pi/4 + 2m\pi\right\rangle$, where m is *any* integer.

Of the possible values of the argument θ of z, the one that satisfies $-\pi < \theta \le \pi$ is called the **principal value of the argument**. So, from Activity 3.4(b), the *principal* value of $\arg(-2-2i)$ is $-3\pi/4$. Complex numbers in polar form are often expressed using the principal value.

Some texts call this θ the 'principal argument' of z.

Complex multiplication in polar form

Using the polar form it is possible to give a simple geometric description of multiplication of complex numbers. Suppose that z has polar form $\langle r, \theta\rangle$ and w has polar form $\langle s, \phi\rangle$. Then what is the polar form of $z \times w$? To find this, we first go back into Cartesian form. We have

$$\begin{aligned} z \times w &= r(\cos\theta + i\sin\theta) \times s(\cos\phi + i\sin\phi) \\ &= rs\cos\theta\cos\phi + (rs\cos\theta\sin\phi)i + (rs\sin\theta\cos\phi)i + (rs\sin\theta\sin\phi)i^2 \\ &= rs\left((\cos\theta\cos\phi - \sin\theta\sin\phi) + (\cos\theta\sin\phi + \sin\theta\cos\phi)i\right). \end{aligned}$$

Now we use the formulas

See Chapter A3, Section 3.

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi,$$

to obtain

$$z \times w = rs(\cos(\theta+\phi) + i\sin(\theta+\phi)).$$

This is just the Cartesian form of the complex number with polar form

$$\langle rs, \theta+\phi\rangle.$$

Hence, to multiply two complex numbers in polar form, you just multiply their moduli and add their arguments. Thus we have an elegant description of the multiplication of complex numbers in terms of their polar forms.

The plural of 'modulus' is 'moduli'.

Multiplication of complex numbers in polar form

If $z = \langle r, \theta\rangle$ and $w = \langle s, \phi\rangle$, then

$$z \times w = \langle rs, \theta+\phi\rangle.$$

Activity 3.5 Multiplication in polar form

Let $z = -1 + i$ and $w = -2 - 2i$. We found polar forms of these earlier:

$$z = \left\langle \sqrt{2}, \frac{3\pi}{4}\right\rangle; \quad w = \left\langle 2\sqrt{2}, \frac{5\pi}{4}\right\rangle.$$

See Example 3.1 and Activity 3.4(b).

Using these polar forms, calculate each of the products below, and express the results in Cartesian form.

(a) $z \times w$ (b) $z \times z$ (c) $z^3 = z \times (z \times z)$

Comment

(a) $z \times w = \left\langle \sqrt{2} \times 2\sqrt{2}, \frac{3\pi}{4} + \frac{5\pi}{4}\right\rangle = \langle 4, 2\pi\rangle = \langle 4, 0\rangle.$

In Cartesian form, this is 4.

(b) $z \times z = \left\langle \sqrt{2} \times \sqrt{2}, \frac{3\pi}{4} + \frac{3\pi}{4}\right\rangle = \left\langle 2, \frac{3\pi}{2}\right\rangle = \left\langle 2, -\frac{\pi}{2}\right\rangle.$

Here $-\pi/2$ is the principal value of the argument, since

$$\frac{3\pi}{2} = 2\pi - \frac{\pi}{2}$$

and

$$-\pi < -\pi/2 \leq \pi.$$

In Cartesian form, this is

$$2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = -2i.$$

(c) $z \times (z \times z) = \left\langle \sqrt{2} \times 2, \frac{3\pi}{4} - \frac{\pi}{2}\right\rangle = \left\langle 2\sqrt{2}, \frac{\pi}{4}\right\rangle.$

In Cartesian form, this is

$$2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 2 + 2i.$$

Activity 3.6 Multiplication by i

Describe geometrically the effect on an arbitrary complex number of multiplying by i.

Comment

The polar form for i is $\langle 1, \pi/2 \rangle$ (see Activity 3.4(a)). Taking a general complex number z with polar form $\langle r, \theta \rangle$, we find that

$$z \times i = \langle r, \theta \rangle \times \left\langle 1, \frac{\pi}{2} \right\rangle = \left\langle r, \theta + \frac{\pi}{2} \right\rangle.$$

The modulus has remained unchanged, but the argument has been increased by $\pi/2$. Thus multiplication by i corresponds to a quarter turn anticlockwise about the origin; see Figure 3.8.

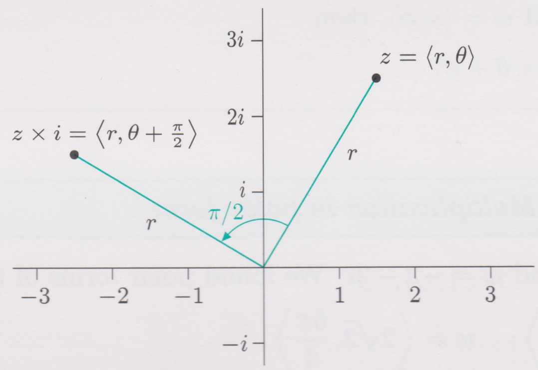


Figure 3.8

The polar form gives us a simple way of calculating powers of complex numbers. Suppose a complex number z has polar form $\langle r, \theta \rangle$. Then

$$z^2 = \langle r, \theta \rangle \times \langle r, \theta \rangle = \langle r \times r, \theta + \theta \rangle = \langle r^2, 2\theta \rangle,$$
$$z^3 = z^2 \times z = \langle r^2, 2\theta \rangle \times \langle r, \theta \rangle = \langle r^2 \times r, 2\theta + \theta \rangle = \langle r^3, 3\theta \rangle,$$

and so on. As we increase the power, the modulus increases by the factor r whereas the argument increases by adding θ. We find the following general result for powers.

In Chapter D4, we show how to prove such a generalization formally.

Powers in polar form

For a complex number $z = \langle r, \theta \rangle$ and $n \in \mathbb{N}$, we have

$$z^n = \langle r^n, n\theta \rangle.$$

Activity 3.7 Calculating a power

See Example 3.1.

We saw earlier that $-1 + i$ has polar form $\langle \sqrt{2}, 3\pi/4 \rangle$. Find

$$(-1 + i)^{10},$$

giving the result in Cartesian form.

Comment

Using the formula for a power, in polar form, we obtain

$$(-1+i)^{10} = \left\langle (\sqrt{2})^{10}, 10 \times \frac{3\pi}{4} \right\rangle = \left\langle 2^5, \frac{15\pi}{2} \right\rangle = \left\langle 32, -\frac{\pi}{2} \right\rangle.$$

Here
$$\frac{15\pi}{2} = 8\pi - \frac{\pi}{2}.$$

In Cartesian form, this is

$$32\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = 32(-i) = -32i.$$

To prove that multiplication of complex numbers is associative using the Cartesian form is messy. In polar form, the proof is much neater.

Activity 3.8 Multiplication is associative

Suppose that z, w and u are any three complex numbers. Working with their polar forms $z = \langle r, \theta \rangle$, $w = \langle s, \phi \rangle$ and $u = \langle t, \psi \rangle$, prove that

$$(z \times w) \times u = z \times (w \times u).$$

A solution is given on page 51.

Summary of Section 3

A complex number can be represented as a point in the plane, known as an Argand diagram. Either Cartesian or polar coordinates may be used, and these correspond to two alternative ways of expressing a complex number, Cartesian form and polar form. The polar form is written $\langle r, \theta \rangle$, where r is the modulus and θ the argument of the complex number. The equivalent Cartesian form is given by

$$\langle r, \theta \rangle = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta).$$

In polar form, we can express multiplication of complex numbers as

$$\langle r, \theta \rangle \times \langle s, \phi \rangle = \langle rs, \theta + \phi \rangle.$$

This leads to a compact expression for a power of a complex number:

$$\langle r, \theta \rangle^n = \langle r^n, n\theta \rangle.$$

Exercises for Section 3

Exercise 3.1

(a) Show the points $z = 2 - i$ and $w = \langle 2, \pi/3 \rangle$ on an Argand diagram.

(b) Convert z to polar form and w to Cartesian form.

(c) What is w^3 in Cartesian form?

Exercise 3.2

(a) For $z = 12 - 5i$, find: (i) $\overline{z}$; (ii) $|z|$; (iii) $z \times \overline{z}$.

(b) For a general complex number z, show that $|z|^2 = z \times \overline{z}$.

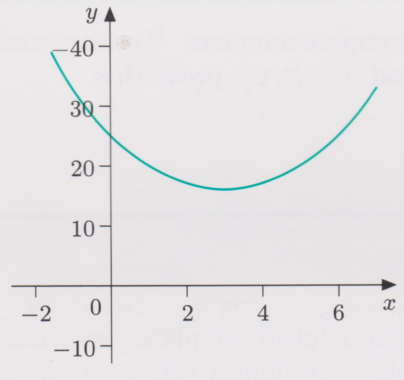
(c) For a general non-zero complex number z, express $1/z$ in terms of its conjugate $\overline{z}$ and modulus $|z|$.

4 Roots of polynomials

4.1 Polynomials

Consider the real function $f(x) = x^2 - 6x + 25$ ($x \in \mathbb{R}$). The solutions of the equation $f(x) = 0$ are referred to as the **roots** of f, and this terminology is extended to other polynomial functions. Now if we look at the graph of this function f (see Figure 4.1(a)), we see that it does not intersect the x-axis. There are no real solutions of the equation

$$x^2 - 6x + 25 = 0.$$



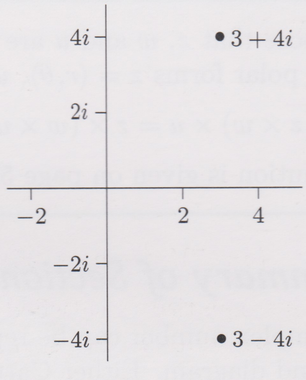


Figure 4.1 (a) The graph of $f(x) = x^2 - 6x + 25$ ($x \in \mathbb{R}$) (b) The complex roots of $z^2 - 6z + 25$ shown on an Argand diagram

However, the equation

$$z^2 - 6z + 25 = 0,$$

where z can take complex values, *does* have solutions. They are $3 + 4i$ and $3 - 4i$. They are not points where a graph cuts an axis, but they can be shown as points on an Argand diagram (see Figure 4.1(b)).

In discussing roots of polynomials, we need to be clear as to whether we are considering just roots that are real, called **real roots**, or are allowing the possibility of roots that are complex numbers, called **complex roots**. Now the real numbers are embedded in the complex numbers, since they are of the form $a + 0i$, where a is real. So if we look for the complex roots of a polynomial, then we shall find any real roots as a special case. The formula for the solution of a quadratic equation

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{4.1}$$

gives all the roots, both real and complex, of a general quadratic polynomial $az^2 + bz + c$. A real root occurs when equation (4.1) gives a complex root z whose imaginary part is 0.

Activity 4.1 Finding roots

(a) Consider the quadratic polynomial $z^2 - 6z + c$, where c is real. For what values of c does this polynomial have real roots?

(b) Multiply out the expression

$$(z + 7)(z^2 - 6z + 25),$$

and hence find the (complex) roots of the polynomial

$$z^3 + z^2 - 17z + 175.$$

Comment

(a) The roots are real when $b^2 - 4ac \geq 0$, which in this case (with $a = 1$, $b = -6$) gives

$$36 - 4c \geq 0, \qquad \text{or equivalently } c \leq 9.$$

(b) We have

$$\begin{aligned}(z + 7)(z^2 - 6z + 25) &= z^3 - 6z^2 + 25z + 7z^2 - 42z + 175\\ &= z^3 + z^2 - 17z + 175.\end{aligned}$$

The roots of $z^3 + z^2 - 17z + 175$ are the solutions of the equation $z^3 + z^2 - 17z + 175 = 0$, that is, of

$$(z + 7)(z^2 - 6z + 25) = 0.$$

This equation is satisfied if $z = -7$ or $z^2 - 6z + 25 = 0$. We can find the solutions of this quadratic equation using the formula. They are

$$z = \frac{6 \pm \sqrt{36 - 100}}{2} = 3 \pm 4i.$$

So the three roots of the cubic polynomial $z^3 + z^2 - 17z + 175$ are -7 and $3 \pm 4i$.

In this section, we shall be concerned with complex roots of polynomials. By a **polynomial** in the variable z, we mean any expression of the form

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \quad (a_n \neq 0). \tag{4.2}$$

Here $a_0, a_1, \ldots, a_n$ are numbers (usually real numbers in our discussion), referred to as **coefficients** of the polynomial. The variable z might be real or complex, but in talking about the roots of a polynomial, we shall mean complex roots. In equation (4.2), n is an integer (with $n \geq 1$), which is called the **degree** of the polynomial. In particular, a *quadratic* is a polynomial of degree 2, a *cubic* is a polynomial of degree 3, a *quartic* is a polynomial of degree 4, and a *quintic* is a polynomial of degree 5.

In a graph such as that in Figure 4.1(a), we are plotting a polynomial in a *real* variable, whereas Figure 4.1(b) shows the complex roots of the same polynomial (now allowing the possibility of the variable being complex). The following video band looks further at the relationship between the graph of a polynomial and its complex roots.

Now watch DVD00115 band D(ii), 'Roots of polynomials'.

Factors and roots

There is a close relationship between the roots of a polynomial and its factors. For example, we have

$$z^2 - 5z + 6 = (z-2)(z-3).$$

The roots of $z^2 - 5z + 6$ can be found from the equation $(z-2)(z-3) = 0$. Either $z - 2 = 0$, in which case $z = 2$, or $z - 3 = 0$, in which case $z = 3$.

This illustrates a general relationship. A number α is a root of the polynomial $p(z)$ if and only if $(z - \alpha)$ is a factor of $p(z)$; that is, $p(z) = (z - \alpha)q(z)$, where $q(z)$ is some other polynomial factor.

In particular, the quadratic equation

$$az^2 + bz + c = 0$$

has solutions α and β if and only if

$$az^2 + bz + c = a(z-\alpha)(z-\beta).$$

This property is true regardless of whether the solutions are real or complex numbers. We have already seen that if the coefficients a, b and c are real but the solutions α and β are not real, then these solutions form a pair of complex conjugates.

See Activity 2.11(c).

In the special case where $b^2 = 4ac$, we say that the polynomial $az^2 + bz + c$ has 'two coincident roots'. This terminology is used to emphasise that the quadratic polynomial is still the product of *two* linear factors,

$$a(z-\alpha)^2,$$

but in this case the two factors are the same. In general, we say that α is a **repeated root** of the polynomial $p(z)$ if $(z-\alpha)^2$ is a factor of $p(z)$.

A very powerful result, proved by Gauss in 1799, states that any polynomial of degree n,

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \quad (a_n \neq 0),$$

can be fully factorised into linear factors

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = a_n(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n),$$

where the n roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ are complex numbers and may involve repetition. Remarkably this result is still true even if the coefficients $a_0, a_1, \ldots, a_n$ are themselves complex numbers. We shall not prove this result here.

Here is an example of such a factorisation.

Activity 4.2 Factorising a cubic

(a) Verify that 1 is a solution of the equation

$$z^3 - 7z^2 + 31z - 25 = 0.$$

(b) Express $z^3 - 7z^2 + 31z - 25$ as $(z-1)$ times a quadratic polynomial.

(c) Fully factorise the cubic polynomial in part (b).

Comment

(a) $1^3 - 7 \times 1^2 + 31 \times 1 - 25 = 0$, and hence 1 *is* a solution.

(b) Writing

$$z^3 - 7z^2 + 31z - 25 = (z-1)(z^2 + bz + c),$$

and expanding the right-hand side, gives

$$z^3 - 7z^2 + 31z - 25 = z^3 + (b-1)z^2 + (c-b)z - c.$$

Comparing coefficients we have

$$b - 1 = -7, \quad c - b = 31 \quad \text{and} \quad -c = -25.$$

Hence $b = -6$ and $c = 25$, and so

$$z^3 - 7z^2 + 31z - 25 = (z-1)(z^2 - 6z + 25).$$

Note that these values for b and c satisfy all three equations.

(c) In Activity 4.1(b) we found the roots of the quadratic factor $z^2 - 6z + 25$ to be $3 \pm 4i$. Thus we can factorise the polynomial completely to give

$$z^3 - 7z^2 + 31z - 25 = (z-1)(z-(3+4i))(z-(3-4i)).$$

For a polynomial of degree n with *real* coefficients, it can be proved that the n roots are either real or come in pairs of complex conjugates. This fact has various implications.

- For a quadratic, we have two roots which are either both real (and possibly the same) or which form a pair of complex conjugates.
- For a cubic, we have three roots which are either all real (with some possibly the same) or one is real and the other two form a pair of complex conjugates. In either case there is at least one real root.

Activity 4.3 Roots of quartics and quintics

(a) What are the possible combinations of real and complex conjugate roots of a quartic or quintic polynomial with real coefficients?

(b) Give an example of a quartic polynomial with real coefficients that has a complex (non-real) root that is repeated?
(*Hint*: It will help to use the result of Activity 2.1(b).)

Solutions are given on page 52.

We have seen that a polynomial of degree n has n roots, if we count repeated roots. We can find these roots in the case of a quadratic, where we have a formula. There are also formulas for finding the roots of cubic and quartic polynomials, although they are very cumbersome. But for quintic polynomials, and those of higher degree, there are no general formulas. However, there are numerical algorithms which can find all the roots to any level of accuracy.

Proved in 1824 by Niels Henrik Abel (1802–29), this negative result requires sophisticated mathematics!

4.2 Roots of unity

The word 'root' is used not only in the context of polynomials but also in the notions of 'square root', 'cube root', and so on. This overlap in terminology is no accident! For example, $z = \sqrt{2}$ is a solution of the equation $z^2 = 2$, or equivalently, $z^2 - 2 = 0$. So the roots of the polynomial $z^2 - 2$ are $\pm\sqrt{2}$, the positive and negative square roots of 2. A cube root, such as $\sqrt[3]{5}$, is a solution of $z^3 = 5$, so it is a root of the polynomial $z^3 - 5$. A similar situation holds for higher-order roots: for example, $\sqrt[5]{10}$ is a root of the polynomial $z^5 - 10$. In general, $\sqrt[n]{a}$ is a root of the polynomial $z^n - a$, for $n \geq 2$.

Now for a quadratic polynomial $z^2 - a$, there are two roots; assuming that $a > 0$, these roots are $\pm\sqrt{a}$. For a cubic polynomial like $z^3 - 8$, there are three roots. One of these is $\sqrt[3]{8} = 2$. But what are the other two?

Activity 4.4 Cube roots of 8

(a) Expand the product $(z - 2)(z^2 + 2z + 4)$.

(b) Find all the complex roots of $z^3 - 8$.

Comment

(a) We obtain

$$(z - 2)(z^2 + 2z + 4) = z^3 + 2z^2 + 4z - 2z^2 - 4z - 8 = z^3 - 8.$$

(b) From part (a), the three roots of $z^3 - 8$ are 2 and the two roots of $z^2 + 2z + 4$. To find these, we can use the formula for the roots of a quadratic, to obtain

$$\frac{-2 \pm \sqrt{2^2 - 4 \times 4}}{2} = -1 \pm i\sqrt{3}.$$

Hence the roots of $z^3 - 8$ are 2 and $-1 \pm i\sqrt{3}$.

So, as well as its real cube root 2, there are two other complex cube roots of 8, namely, $-1 + i\sqrt{3}$ and $-1 - i\sqrt{3}$. Once we allow complex roots, we find other cube roots (and fourth and fifth roots, and so on) besides the familiar real ones.

A polynomial of the form $z^n - 1$ is of degree n, and so it has n roots. Thus even the number 1 has n nth roots. These are referred to as the **nth roots of unity**. But how can we find them? There is a neat way of doing this, which we shall discuss shortly. But first, let us see how to check whether a given complex number is a root of unity.

Activity 4.5 Checking roots of unity

(a) Show that i is a fourth root of unity.

(b) Show that the complex number with polar form $\langle 1, 4\pi/3 \rangle$ is a cube root of unity.

Comment

(a) To be a fourth root of unity, i must satisfy the equation $z^4 = 1$. That is, the fourth power of i should be 1. Well

$$i^4 = (i^2)^2 = (-1)^2 = 1, \qquad \text{as required.}$$

(b) To be a cube root of unity, $\langle 1, 4\pi/3\rangle$ must be a root of $z^3 - 1$. That is, the cube of $\langle 1, 4\pi/3\rangle$ should be 1. Well

$$\left\langle 1, \frac{4\pi}{3}\right\rangle^3 = \left\langle 1^3, 3 \times \frac{4\pi}{3}\right\rangle = \langle 1, 4\pi\rangle = \langle 1, 0\rangle,$$

since 4π is an integer multiple of 2π. Since $\langle 1, 0\rangle$ is the polar form of the real number 1, we have shown that $z = \langle 1, 4\pi/3\rangle$ does satisfy the equation $z^3 = 1$.

The next example shows you how to *find* roots of unity.

Example 4.1 Cube roots of unity

Solve $z^3 = 1$ to find the three cube roots of unity, and show them on an Argand diagram.

Solution

First we represent z by $\langle r, \theta\rangle$ and 1 by $\langle 1, 0\rangle$. The equation $z^3 = 1$ becomes $\langle r, \theta\rangle^3 = \langle 1, 0\rangle$ and then $\langle r^3, 3\theta\rangle = \langle 1, 0\rangle$. From this we see that $r^3 = 1$ and $3\theta = 2m\pi$, where m is some integer. This leads to $r = 1$ (remember r is real and positive) and $\theta = 2m\pi/3$. Hence the general solution of $z^3 = 1$ is

$$z = \left\langle 1, \frac{2m\pi}{3}\right\rangle.$$

The three distinct roots are found by setting $m = 0$, 1 and 2. Converting to Cartesian form we have, in general,

$$\left\langle 1, \frac{2m\pi}{3}\right\rangle = \cos\left(\frac{2m\pi}{3}\right) + i\sin\left(\frac{2m\pi}{3}\right). \tag{4.3}$$

When $m = 0$ this gives

$$\cos(0) + i\sin(0) = 1.$$

When $m = 1$ this gives

$$\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

When $m = 2$ this gives

$$\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Putting other values of m (in $\mathbb{Z}$) into equation (4.3) simply gives repetitions of these three roots. They are shown on an Argand diagram in Figure 4.2 overleaf.

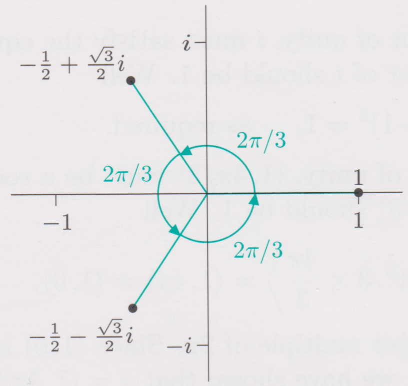


Figure 4.2 The cube roots of unity

Here is another example.

Activity 4.6 Fifth roots of unity

Find the fifth roots of unity. Give them in Cartesian form, and show them on an Argand diagram.

Comment

The fifth roots of unity are roots of $z^5 - 1$, and so they are solutions of $z^5 = 1$. Representing z in polar form, as $\langle r, \theta \rangle$, and 1 in polar form, as $\langle 1, 0 \rangle$, we obtain

$$\langle r, \theta \rangle^5 = \langle 1, 0 \rangle; \quad \text{that is,} \quad \langle r^5, 5\theta \rangle = \langle 1, 0 \rangle.$$

Therefore

$$r^5 = 1 \quad \text{and} \quad 5\theta = 2m\pi,$$

where $m \in \mathbb{Z}$. Hence $r = 1$ and the five roots correspond to the following five values of θ obtained by setting $m = 0, 1, 2, 3$ and 4:

$$0, \quad \frac{2\pi}{5}, \quad \frac{4\pi}{5}, \quad \frac{6\pi}{5}, \quad \frac{8\pi}{5}.$$

(Other values of $m \in \mathbb{Z}$ simply repeat one of the roots below.) So the fifth roots of unity are z_0, z_1, z_2, z_3 and z_4, where:

$$z_0 = \langle 1, 0 \rangle = 1,$$

$$z_1 = \left\langle 1, \frac{2\pi}{5} \right\rangle = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) = 0.3090 + 0.9511i,$$

$$z_2 = \left\langle 1, \frac{4\pi}{5} \right\rangle = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) = -0.8090 + 0.5878i,$$

$$z_3 = \left\langle 1, \frac{6\pi}{5} \right\rangle = \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right) = -0.8090 - 0.5878i,$$

$$z_4 = \left\langle 1, \frac{8\pi}{5} \right\rangle = \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right) = 0.3090 - 0.9511i,$$

to four decimal places. The roots are spaced equally around the unit circle, giving the Argand diagram in Figure 4.3.

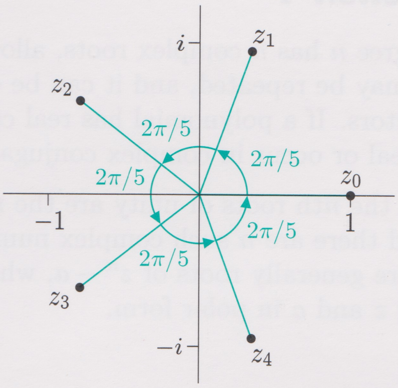


Figure 4.3 The fifth roots of unity

We can find nth roots of *any* complex number a using a similar approach. To solve the equation $z^n = a$, we represent z and a in polar form as $\langle r, \theta\rangle$ and $\langle s, \phi\rangle$, respectively. The equation $z^n = a$ becomes $\langle r, \theta\rangle^n = \langle s, \phi\rangle$, which is equivalent to $\langle r^n, n\theta\rangle = \langle s, \phi\rangle$. The two parts of this equation give $r^n = s$ and $n\theta = \phi + 2m\pi$, where m is any integer. These are solved directly for r (which must be real and positive) and for θ, with m taking the values $0, 1, \ldots, n-1$. We obtain the polar form $\langle r, \theta\rangle$ for each of the roots, and then we find their Cartesian forms, using $r\cos\theta + ir\sin\theta$.

Note that the n roots are always regularly spaced around a circle centred at 0.

Activity 4.7 Fourth roots of 4i

Find the four fourth roots of $4i$ by solving

$$z^4 = 4i.$$

Comment

Let $z = \langle r, \theta\rangle$. Since $4i = \langle 4, \pi/2\rangle$, the equation $z^4 = 4i$ becomes

$$\langle r^4, 4\theta\rangle = \left\langle 4, \frac{\pi}{2}\right\rangle.$$

For the modulus we have $r^4 = 4$, which gives $r = \sqrt{2}$.

For the argument we have $4\theta = \pi/2 + 2m\pi$, where $m = 0, 1, 2, 3$. This gives

$$\theta = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8} \text{ and } \frac{13\pi}{8}.$$

So the fourth roots of $4i$ are:

$$\left\langle \sqrt{2}, \frac{\pi}{8}\right\rangle = \sqrt{2}\cos\left(\frac{\pi}{8}\right) + i\sqrt{2}\sin\left(\frac{\pi}{8}\right) = 1.307 + 0.541i,$$

$$\left\langle \sqrt{2}, \frac{5\pi}{8}\right\rangle = \sqrt{2}\cos\left(\frac{5\pi}{8}\right) + i\sqrt{2}\sin\left(\frac{5\pi}{8}\right) = -0.541 + 1.307i,$$

$$\left\langle \sqrt{2}, \frac{9\pi}{8}\right\rangle = \sqrt{2}\cos\left(\frac{9\pi}{8}\right) + i\sqrt{2}\sin\left(\frac{9\pi}{8}\right) = -1.307 - 0.541i,$$

$$\left\langle \sqrt{2}, \frac{13\pi}{8}\right\rangle = \sqrt{2}\cos\left(\frac{13\pi}{8}\right) + i\sqrt{2}\sin\left(\frac{13\pi}{8}\right) = 0.541 - 1.307i,$$

to three decimal places.

Notice that these roots do not occur in complex conjugate pairs. They are the roots of a polynomial, $z^4 - 4i$, that does not have *real* coefficients.

Summary of Section 4

Any polynomial of degree n has n complex roots, allowing for the possibility that some may be repeated, and it can be expressed as a product of n linear factors. If a polynomial has real coefficients, then its roots must either be real or occur in complex conjugate pairs.

For any integer $n \geq 2$, the nth roots of unity are the roots of the polynomial $z^n - 1$, and there are n such complex numbers. We can find roots of unity, and more generally roots of $z^n - a$, where a is any complex number, by expressing z and a in polar form.

Exercises for Section 4

Exercise 4.1

Give an example of a polynomial which has the four roots $1 \pm 2i$ and $2 \pm i$.

Exercise 4.2

Explain why a ninth-order polynomial with real coefficients must have at least one real root.

Exercise 4.3

Find the three cube roots of -1 and show them on an Argand diagram.

5 Complex exponentials

In this section we shall introduce a final operation on complex numbers. What might e^z mean, where z is a complex number? We shall define this, and see how such complex exponentials provide an elegant description of spirals.

5.1 Euler's formula

We start by looking at complex exponentials of the form $e^{i\theta}$.

Recall that the Taylor series about 0 for e^x is

See Chapter C3, Section 3 for the Taylor series in this discussion.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad (x \in \mathbb{R}).$$

Suppose that we substitute $x = i\theta$ into this series. This suggests that

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots . \qquad (5.1) \end{aligned}$$

We have

$$i^2 = -1,$$
$$i^3 = -i,$$
$$i^4 = 1,$$
$$i^5 = i,$$

and so on.

The real terms in this series are

$$1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots .$$

This is the Taylor series about 0 for $\cos\theta$.

The real multiples of i in equation (5.1) are

$$i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right).$$

The terms in the brackets form the Taylor series about 0 for $\sin\theta$.

Together, these expressions suggest that

$$e^{i\theta} = \cos\theta + i\sin\theta. \qquad (5.2)$$

The argument leading to equation (5.2) is certainly not a proof! It does not address the meaning of infinite series containing complex terms, nor the validity of separating and regrouping terms in an infinite series, as we have done. But it is a persuasive argument, and based on it we shall use equation (5.2) as a *definition* of $e^{i\theta}$. Equation (5.2) is known as **Euler's formula**.

Euler introduced this fundamental formula in 1748.

Activity 5.1 Exponential roots

For $\theta = \pi/4$, give each of the eight complex numbers $e^{ni\theta}$ $(n = 0, 1, 2, \ldots, 7)$ in Cartesian form, and mark them on an Argand diagram.

Comment

This is as we would hope, since $a^0 = 1$ for any non-zero real number a.

With $n = 0$ we obtain $\quad e^0 = \cos(0) + i\sin(0) = 1.$

With $n = 1$ we obtain $\quad e^{i\pi/4} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(1+i).$

With $n = 2$ we obtain $\quad e^{i\pi/2} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i.$

With $n = 3$ we obtain $\quad e^{3i\pi/4} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}(-1+i).$

With $n = 4$ we obtain $\quad e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1.$

With $n = 5$ we obtain $\quad e^{5i\pi/4} = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}}(-1-i).$

With $n = 6$ we obtain $\quad e^{3i\pi/2} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i.$

With $n = 7$ we obtain $\quad e^{7i\pi/4} = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}}(1-i).$

These complex numbers are shown on an Argand diagram in Figure 5.1.

The regular spacing of these complex numbers suggests that they are the eighth roots of unity. We would expect that this is the case, since (assuming that complex exponentials obey the rule $(e^a)^b = e^{a\times b}$ that holds for real numbers a and b) we have, for any integer n,

$$\left(e^{ni\pi/4}\right)^8 = e^{8ni\pi/4} = e^{2ni\pi} = \cos(2n\pi) + i\sin(2n\pi) = 1.$$

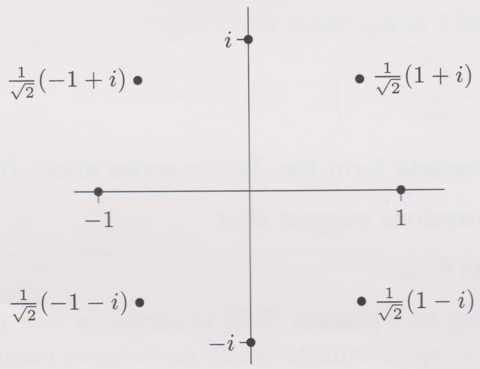


Figure 5.1 $\;e^{ni\theta}$ with $\theta = \pi/4$ and $n = 0, 1, 2, \ldots, 7$

Euler's formula gives a definition of e^z in the case where z is a real multiple of i. Since we expect that $e^{a+bi} = e^a \times e^{bi}$, we make the following definition.

Complex exponentials

Let $z = a + bi$. Then

$$e^z = e^a \times e^{bi} = e^a(\cos b + i\sin b). \tag{5.3}$$

Here is a particular complex exponential to evaluate.

Activity 5.2 A complex exponential

Express e^{2+3i} in Cartesian form.

Comment

Using equation (5.3), we obtain

$$e^{2+3i} = e^2(\cos 3 + i \sin 3) = -7.315 + 1.043i.$$

The expression on the right-hand side of Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

is already familiar from our discussion of the relationship between the polar and Cartesian forms for a complex number. We know that

$$\langle r, \theta\rangle = r(\cos\theta + i\sin\theta).$$

Comparing these two formulas we find that

$$\langle r, \theta\rangle = re^{i\theta}.$$

So our definition of complex exponentials gives us another representation of complex numbers, namely as $re^{i\theta}$, which we call the **exponential form**.

Activity 5.3 Exponential form for complex numbers

Put the complex number $1 + i$ in polar form and then in exponential form.

Comment

The complex number $1 + i$ has modulus $\sqrt{2}$ and argument $\pi/4$, and hence has polar form $\left\langle \sqrt{2}, \pi/4\right\rangle$. This transforms directly into the exponential form $\sqrt{2}e^{i\pi/4}$. That is,

$$1 + i = \left\langle \sqrt{2}, \frac{\pi}{4}\right\rangle = \sqrt{2}e^{i\pi/4}.$$

We now have three different representations of complex numbers. Each is useful in its own way and each throws light on the others. We can use the fact that $e^{a+bi} = e^a(\cos b + i\sin b)$ has polar form $\langle e^a, b\rangle$ to verify that complex exponentials have various properties that we would expect, by analogy with real exponentials.

Activity 5.4 Properties of e^z

Let $z = a + bi$ and $w = c + di$ be any two complex numbers.

(a) Give, in polar form, each of:

(i) $e^z \times e^w$; (ii) e^{z+w};

and show that they are equal.

(b) Express e^{-z} in polar form. Calculate $e^z \times e^{-z}$, and show that it is 1.

Solutions are given on page 52.

For real exponentials, we have $e^x \times e^y = e^{x+y}$.

For any real number x, e^{-x} means $1/e^x$, so $e^x \times e^{-x} = 1$.

5.2 Circles and spirals

This subsection will not be assessed.

Here, we shall look at sequences generated by recurrences of the form

$$c_0 = 1, \quad c_{n+1} = kc_n \quad (n = 0, 1, 2, \ldots), \tag{5.4}$$

where k is a complex number. Such a recurrence sequence has the equivalent closed form

$$c_n = k^n \quad (n = 0, 1, 2, \ldots). \tag{5.5}$$

See MST121 Chapter A1, where sequences of this form, with k real, are called *geometric sequences*.

What will such a sequence look like, plotted on an Argand diagram?

In Activity 5.1 you saw a sequence generated by a recurrence of the type in equation (5.4). There, in effect, we started with 1 and multiplied repeatedly by $k = e^{i\pi/4}$. We only looked at the first few values in the sequence, but for that value of k we have $k^8 = 1$, and hence $c_8 = c_0$, $c_9 = c_1$, and so on. The subsequent values in the sequence are just repeats of the eight points shown in Figure 5.1.

Activity 5.5 Which recurrences repeat themselves?

For which complex numbers k will the sequence generated by the recurrence

$$c_0 = 1, \quad c_{n+1} = kc_n \quad (n = 0, 1, 2, \ldots),$$

eventually repeat?

Comment

The values in the sequence will eventually repeat if, for some natural number n, we have $k^n = 1$. Then $k^{n+1} = k$, $k^{n+2} = k^2$, and so on, and the finite sequence

$$1, k, k^2, k^3, \ldots, k^{n-1}$$

will be repeated. Now $k^n = 1$ means that k is an nth root of unity. So the sequence repeats if and only if k is a root of unity.

We next consider a recurrence of the type in equation (5.4) for a complex number whose modulus is greater than 1. For convenience, we take a complex number in polar form, $k = \langle 1.1, \pi/6 \rangle$. Then, by equation (5.5),

$$c_n = k^n = \left\langle 1.1, \frac{\pi}{6} \right\rangle^n = \left\langle (1.1)^n, \frac{n\pi}{6} \right\rangle.$$

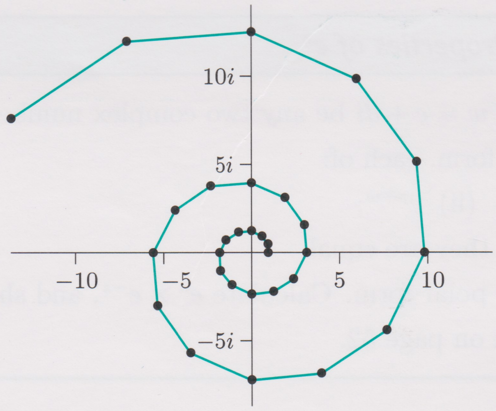


Figure 5.2 The sequence k^n, with $k = \langle 1.1, \pi/6 \rangle$

Figure 5.2 shows the first 30 points in this recurrence sequence on an Argand diagram, connected by line segments. We obtain a curve spiralling outwards, giving a shape suggesting that of a snail or shellfish. Since each step covers an angle of $\pi/6$, it takes 12 steps to go once round the origin. As this happens, the distance from the curve to the origin increases by a factor of $(1.1)^{12}$. This ratio holds for *any* twelve steps in the sequence since, for any natural number n,

$$\frac{|c_{n+12}|}{|c_n|} = \frac{(1.1)^{n+12}}{(1.1)^n} = (1.1)^{12}.$$

So any radial 'spoke' from the origin intersects the spiral at points whose distances from the origin increase geometrically, in the ratio $(1.1)^{12}$. Therefore these distances form a geometric sequence, with common ratio $(1.1)^{12} = 3.138$ to three decimal places.

Activity 5.6 Other spirals

Describe the pattern obtained on an Argand diagram by plotting the sequence

$$c_n = k^n \quad (n = 0, 1, 2, \ldots)$$

with each of the values of k in parts (a)–(d).

(a) $k = \left\langle \sqrt{2}, \frac{\pi}{2} \right\rangle$ (b) $k = \left\langle \frac{1}{\sqrt{2}}, \frac{\pi}{2} \right\rangle$

(c) $k = \left\langle \sqrt{2}, -\frac{\pi}{2} \right\rangle$ (d) $k = \left\langle 1, \frac{\pi}{2} \right\rangle$

Comment

(a) With angle $\pi/2$, the sequence spirals anticlockwise around the origin a quarter turn at a time. One complete rotation of 2π takes 4 steps and increases the distance from the origin by a factor of $(\sqrt{2})^4 = 4$.

(b) Again, with angle $\pi/2$, we spiral anticlockwise around the origin a quarter turn at a time. One complete rotation again takes 4 steps but this time *decreases* the distance from the origin by a factor of $(1/\sqrt{2})^4 = 1/4$ in each rotation. So this time we spiral *in* towards the origin.

(c) With angle $-\pi/2$, we again spiral around the origin a quarter turn at a time, but now in a clockwise direction. With scale factor $\sqrt{2}$, as in part (a), we spiral outwards by a factor of $(\sqrt{2})^4 = 4$ in each complete rotation.

(d) We have $\langle 1, \pi/2 \rangle = i$ so this sequence just repeats the following four values:

$$1, \quad i, \quad -1, \quad -i.$$

The spirals in parts (a), (b) and (c) of Activity 5.6 are shown in Figure 5.3.

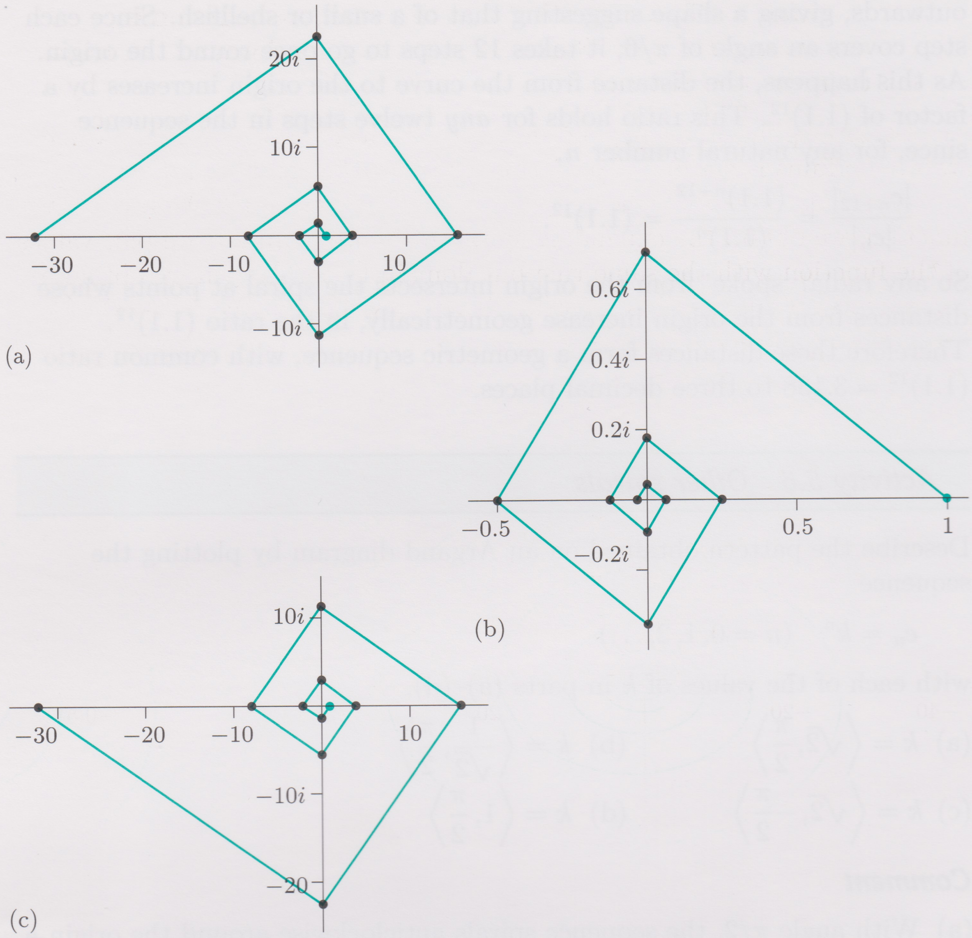


Figure 5.3 (a) $k = \langle\sqrt{2}, \pi/2\rangle$ (b) $k = \langle 1/\sqrt{2}, \pi/2\rangle$ (c) $k = \langle\sqrt{2}, -\pi/2\rangle$

Continuous spirals

The sequence $c_n = k^n$ $(n = 0, 1, 2, \ldots)$ gives a plot of a discrete spiral. To produce a continuous spiral on an Argand diagram, we need a function with a continuous parameter.

Parametrisation functions were introduced in Chapter A3, Section 1.

First recall that the parametrisation function $f : \mathbb{R} \to \mathbb{R}^2$, where $f(t) = (\cos t, \sin t)$, has the unit circle in $\mathbb{R}^2$ as its image set. As the value of the parameter t varies in $\mathbb{R}$, the point $(x, y) = (\cos t, \sin t)$ travels round the unit circle; see Figure 5.4.

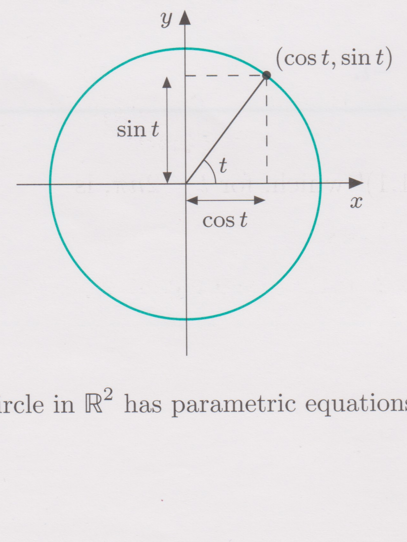


Figure 5.4 The unit circle in $\mathbb{R}^2$ has parametric equations $x = \cos t, y = \sin t$

Now we can obtain the same curve on an Argand diagram if we plot values of $\cos t + i\sin t = e^{it}$. More generally, plotting re^{it}, where $r > 0$, produces a circle of radius r. The corresponding parametrisation function is $g : \mathbb{R} \to \mathbb{C}$, where $g(t) = re^{it}$. Such a function, with codomain $\mathbb{C}$, is referred to as a **complex-valued function**.

To obtain a spiral on an Argand diagram, we need to plot complex values whose modulus is varying. To do this, consider the complex-valued function $f : [0, \infty) \to \mathbb{C}$, where $f(t) = (1.1)^t e^{it}$. Plotting the values of this function, we obtain the expanding spiral in Figure 5.5(a). If we plot values of the function with the same rule but domain $(-\infty, 0]$, then we obtain a contracting spiral; see Figure 5.5(b). If we take the domain to be all of $\mathbb{R}$, then we obtain a 'doubly infinite' spiral, both outwards and inwards.

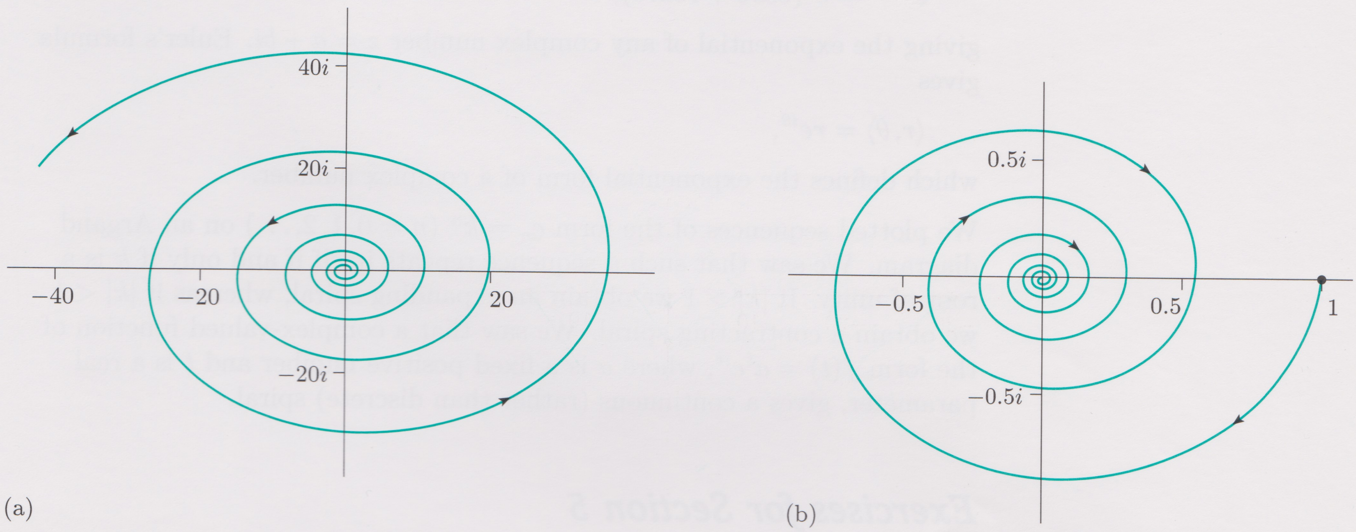


Figure 5.5 Plots of $f(t) = (1.1)^t e^{it}$ for real t, with: (a) $t \geq 0$; (b) $t \leq 0$

Once again, any radial spoke cuts this spiral in points whose distances from the origin form a geometric sequence.

Activity 5.7 Where a spoke cuts the spiral

Find the real values of $t \geq 0$ for which $z = (1.1)^t e^{it}$ lies on the positive real axis. What is $|z|$ for such values of t?

Comment

In polar form, the complex number $z = (1.1)^t e^{it}$ is

$$z = \langle (1.1)^t, t \rangle.$$

Now z lies on the positive real axis if its argument, t, is 0 or a positive integer multiple of 2π; that is, if $t = 2n\pi$, where $n \geq 0$ is an integer.

The modulus of z is $(1.1)^t$ which, for $t = 2n\pi$, is

$$(1.1)^{2n\pi} = ((1.1)^{2\pi})^n = k^n,$$

where $k = (1.1)^{2\pi} = 1.82$ to three significant figures.

This is a geometric sequence with common ratio k.

We obtain similar spirals if we consider any complex-valued function of the general form $f(t) = a^t e^{it}$, where a is a positive constant. We can obtain spirals for which the distance from the origin varies in a different way by considering functions of the form $f(t) = r(t)e^{it}$, with a different type of real function $r(t)$, such as $r(t) = t$ ($t \in [0, \infty)$) for example.

Summary of Section 5

We used Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

to define the exponential of a real multiple of i, and extended this to

$$e^{a+bi} = e^a(\cos b + i\sin b),$$

giving the exponential of any complex number $z = a + bi$. Euler's formula gives

$$\langle r, \theta\rangle = re^{i\theta},$$

which defines the exponential form of a complex number.

We plotted sequences of the form $c_n = k^n$ ($n = 0, 1, 2, \ldots$) on an Argand diagram. We saw that such a sequence repeats itself if and only if k is a root of unity. If $|k| > 1$ we obtain an expanding spiral, whereas if $|k| < 1$ we obtain a contracting spiral. We saw that a complex-valued function of the form $f(t) = a^t e^{it}$, where a is a fixed positive number and t is a real parameter, gives a continuous (rather than discrete) spiral.

Exercises for Section 5

Exercise 5.1

Express $e^{-1+i\pi/3}$ in Cartesian form.

6 Complex numbers and Mathcad

In this section you will see how to use Mathcad to manipulate complex numbers, as described in the earlier sections. Mathcad can be used to:

- ◇ perform calculations with complex numbers and find $|z|$, $\arg(z)$ and $\overline{z}$;
- ◇ investigate roots of polynomials, in particular roots of unity;
- ◇ work with complex exponentials.

Refer to Computer Book D for the work in this section.

Summary of Section 6

You saw how to use Mathcad to manipulate complex numbers. No new mathematics was introduced in this section.

Summary of Chapter D1

The individual section summaries provide a summary of the chapter.

Learning outcomes

You have been working towards the following learning outcomes.

Notation to know and use

$\mathbb{C}$, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $|z|$, $\overline{z}$, $\arg(z)$, $\langle r, \theta \rangle$, $e^{i\theta}$.

Terms to know and use

Complex number. Real and imaginary parts of a complex number. Modulus and argument of a complex number. Complex conjugate. Cartesian, polar and exponential forms of a complex number. Argand diagram. Commutative, associative and distributive operations. Polynomial. Quadratic, cubic, quartic and quintic polynomials. Degree of a polynomial. Root of a polynomial. Root of unity.

Mathematical skills

◇ Perform arithmetic operations on complex numbers: addition, subtraction, multiplication and division.

◇ Switch between the Cartesian, polar and exponential forms of a complex number; in particular, use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

◇ Interpret an Argand diagram. Plot complex numbers on an Argand diagram.

◇ Use the rules for multiplying complex numbers, and for finding powers, in polar and exponential forms.

◇ Calculate roots of unity, and more generally the nth roots of any given complex number.

◇ Describe the plots obtained on an Argand diagram from:

  a sequence $c_n = k^n$ $(n = 0, 1, 2, \ldots)$, for a fixed complex number k;

  a function $f(t) = a^t e^{it}$, for a fixed positive number a.

Ideas to be aware of

Any polynomial in z of order n can be factorised into n linear factors of the form $(z - \alpha)$, where α is a complex number. If a polynomial has real coefficients, then its roots are either real or occur in complex conjugate pairs.

Mathcad skills

Input complex numbers into Mathcad, and perform manipulations with them.

Solutions to Activities

Solution 2.1

(a)
$$\mathrm{Re}(2i) = \mathrm{Re}(0 + 2i) = 0$$
$$\mathrm{Im}(2i) = \mathrm{Im}(0 + 2i) = 2$$
$$\mathrm{Re}(-3) = \mathrm{Re}(-3 + 0i) = -3$$
$$\mathrm{Im}(-3) = \mathrm{Im}(-3 + 0i) = 0$$

(b) The formula for the solution of a quadratic equation gives

$$z = \frac{6 \pm \sqrt{-64}}{2} = \frac{6 \pm \sqrt{64}\sqrt{-1}}{2}$$
$$= \frac{6 \pm 8i}{2} = 3 \pm 4i.$$

So $z_1 = 3 + 4i$, with $\mathrm{Re}(z_1) = 3$ and $\mathrm{Im}(z_1) = 4$, whereas $z_2 = 3 - 4i$, with $\mathrm{Re}(z_2) = 3$ and $\mathrm{Im}(z_2) = -4$.

Solution 2.3

(a) We have

$$\begin{aligned}&(-3 + 2i) \times (6 - 5i)\\ &\quad = -18 + 15i + 12i - 10i^2\\ &\quad = -18 + 27i + 10\\ &\quad = -8 + 27i.\end{aligned}$$

(You could obtain the same result by substituting in the formula for a general product.)

(b) We have

$$\begin{aligned}&(a + bi) \times (c + di)\\ &\quad = a \times c + (a \times d)i + (b \times c)i + (b \times d)i^2\\ &\quad = ac + adi + bci - bd\\ &\quad = (ac - bd) + (ad + bc)i,\end{aligned}$$

as is given in the definition of multiplication.

Solution 2.5

We need to show that, for any complex numbers $z = a + bi$ and $w = c + di$,

$$z \times w = w \times z.$$

From the definition of multiplication of complex numbers, we have:

$$z \times w = (ac - bd) + (ad + bc)i;$$
$$w \times z = (ca - db) + (cb + da)i.$$

Since

$$ac - bd = ca - db \quad \text{and} \quad ad + bc = cb + da,$$

we can see that the complex numbers $z \times w$ and $w \times z$ have equal real parts and equal imaginary parts, so $z \times w = w \times z$. Hence multiplication of complex numbers is commutative.

Solution 2.6

Remember that $0 = 0 + 0i$ and $1 = 1 + 0i$. Let $z = a + bi$. We have

$$\begin{aligned}z + 0 &= (a + 0) + (b + 0)i = a + bi = z;\\ 0 \times z &= (0 \times a - 0 \times b) + (0 \times b + 0 \times a)i\\ &= 0 + 0i = 0;\\ 1 \times z &= (1 \times a - 0 \times b) + (1 \times b + 0 \times a)i\\ &= a + bi = z.\end{aligned}$$

Solution 2.7

We have

$$\begin{aligned}z - w &= (-3 + 4i) - (12 - 5i)\\ &= (-3 - 12) + (4 + 5i)i\\ &= -15 + 9i.\end{aligned}$$

Solution 3.1

See Figure S.1.

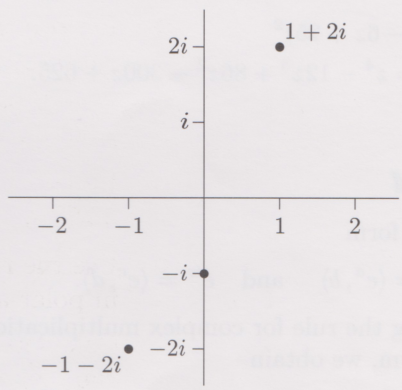


Figure S.1

Solution 3.8

The polar forms are

$$z = \langle r, \theta \rangle, \quad w = \langle s, \phi \rangle \quad \text{and} \quad u = \langle t, \psi \rangle.$$

Then

$$z \times w = \langle rs, \theta + \phi \rangle,$$

so

$$(z \times w) \times u = \langle (rs)t, (\theta + \phi) + \psi \rangle;$$

whereas

$$w \times u = \langle st, \phi + \psi \rangle,$$

so

$$z \times (w \times u) = \langle r(st), (\theta + (\phi + \psi)) \rangle.$$

Hence, using associativity of addition and multiplication for real numbers, we have

$$(z \times w) \times u = z \times (w \times u).$$

Solution 4.3

(a) For a quartic polynomial with real coefficients, there are three possibilities. We can have four real roots, or two real roots and a pair of complex conjugate roots, or two pairs of complex conjugate roots.

For a quintic polynomial with real coefficients, we must have at least one real root and then the remaining four roots can be as described above for a quartic.

(b) Consider $(z^2 - 6z + 25)^2$, which is a quartic polynomial with real coefficients.

In Solution 2.1(b) you saw that the equation $z^2 - 6z + 25 = 0$ has a pair of complex conjugate solutions $3 \pm 4i$, so

$$z^2 - 6z + 25 = (z - (3 + 4i)(z - (3 - 4i)).$$

Therefore

$$(z^2 - 6z + 25)^2 = (z - (3 + 4i))^2(z - (3 - 4i))^2.$$

So each of the complex conjugates $3 \pm 4i$ is a repeated root of the quartic polynomial.

$$\begin{aligned}&(z^2 - 6z + 25)^2\\ &\quad = z^4 - 12z^3 + 86z^2 - 300z + 625.\end{aligned}$$

Solution 5.4

(a) In polar form

$$e^z = \langle e^a, b\rangle \quad \text{and} \quad e^w = \langle e^c, d\rangle.$$

(i) Using the rule for complex multiplication in polar form, we obtain

$$\begin{aligned} e^z \times e^w &= \langle e^a \times e^c, b + d\rangle \\ &= \langle e^{a+c}, b + d\rangle. \end{aligned}$$

(Since a and c are real, we know that $e^a \times e^c = e^{a+c}$.)

(ii) We have $z + w = (a + c) + (b + d)i$. So, in polar form,

$$e^{z+w} = \langle e^{a+c}, b + d\rangle.$$

Thus we do have

$$e^z \times e^w = e^{z+w}.$$

(b) We have $-z = -a - bi$. So, in polar form,

$$e^{-z} = \langle e^{-a}, -b\rangle.$$

Using the rule for complex multiplication in polar form, we obtain

$$\begin{aligned} e^z \times e^{-z} &= \langle e^a \times e^{-a}, b + (-b)\rangle \\ &= \langle e^{a-a}, 0\rangle \\ &= \langle e^0, 0\rangle \\ &= \langle 1, 0\rangle, \end{aligned}$$

which is the polar form of 1. Thus we do have

$$e^z \times e^{-z} = 1.$$

(Alternatively, by part (a) with $w = -z$, we have

$$\begin{aligned} e^z \times e^{-z} &= e^{z+(-z)} \\ &= e^0 \\ &= 1.) \end{aligned}$$

Solutions to Exercises

Solution 2.1

(a) $3 - i$

(b) $-1 + 2i$

(c) $1 + 3i$

(d) $-4 - 2i$

(e) $4 - 3i$

(f) $\dfrac{1+2i}{5}$

(g) $\dfrac{2+i}{1-2i} = \dfrac{(2+i)(1+2i)}{(1-2i)(1+2i)} = \dfrac{5i}{5} = i$

Solution 2.2

We have

$$\overline{z} = a - bi \quad \text{and} \quad \overline{w} = c - di.$$

So

$$\overline{z} + \overline{w} = (a+c) - (b+d)i.$$

Now

$$z + w = (a+c) + (b+d)i.$$

So

$$\overline{z+w} = (a+c) - (b+d)i.$$

Thus we do have

$$\overline{z} + \overline{w} = \overline{z+w}.$$

Solution 3.1

(a) The Argand diagram is given in Figure S.2.

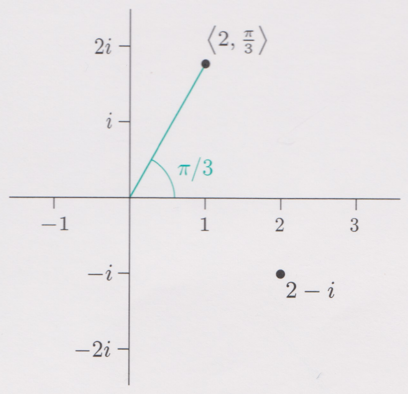


Figure S.2

(b) We have $|z| = \sqrt{2^2 + 1^2} = \sqrt{5}$ and $\arccos(2/\sqrt{5}) = 0.4636$ to four decimal places. Since z lies in the fourth quadrant, the principal value of the argument is -0.4636, so

$$z = \left\langle \sqrt{5}, -0.4636 \right\rangle.$$

The Cartesian form of w is

$$w = 2\cos\left(\frac{\pi}{3}\right) + 2i\sin\left(\frac{\pi}{3}\right) = 1 + i\sqrt{3}.$$

(c) Using the polar form of w, we obtain

$$w^3 = \left\langle 2, \frac{\pi}{3} \right\rangle^3 = \left\langle 2^3, \frac{3\pi}{3} \right\rangle = \langle 8, \pi \rangle = -8.$$

Solution 3.2

(a) We find:

(i) $\overline{z} = 12 + 5i$;

(ii) $|z| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$;

(iii) $z \times \overline{z} = 12^2 + 5^2 = 169$.

(Notice that $|z|^2 = z \times \overline{z}$.)

(b) If $z = a + bi$, then $\overline{z} = a - bi$, and

$$z \times \overline{z} = (a + bi) \times (a - bi) = a^2 + b^2 = |z|^2.$$

(c) If we multiply both the top and the bottom of the fraction $1/z$ by $\overline{z}$ (as we would do to simplify this fraction for a particular complex number z), then we obtain

$$\frac{1}{z} = \frac{\overline{z}}{z \times \overline{z}} = \frac{\overline{z}}{|z|^2},$$

using the result of part (b).

Solution 4.1

A suitable polynomial is

$$(z - (1+2i))(z - (1-2i))(z - (2+i))(z - (2-i)).$$

Since

$$(z - (1+2i))(z - (1-2i)) = z^2 - 2z + 5$$

and

$$(z - (2+i))(z - (2-i)) = z^2 - 4z + 5,$$

the above polynomial is

$$\begin{aligned}&(z^2 - 2z + 5)(z^2 - 4z + 5)\\ &\quad = z^4 - 6z^3 + 18z^2 - 30z + 25.\end{aligned}$$

Solution 4.2

A ninth-order polynomial has nine roots. Any complex roots occur in conjugate pairs, since the polynomial has real coefficients. So there is an even number of non-real roots, and so there can be eight at most. Hence there has to be at least one real root.

Solution 4.3

Let $z = \langle r, \theta \rangle$. In polar coordinates the equation $z^3 = -1$ becomes

$$\langle r^3, 3\theta \rangle = \langle 1, \pi \rangle.$$

Thus we have $r = 1$ and $3\theta = \pi + 2m\pi$, giving

$$\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}.$$

The corresponding solutions are:

$$z_0 = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i;$$

$$z_1 = \cos(\pi) + i\sin(\pi) = -1;$$

$$z_2 = \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

The Argand diagram is shown in Figure S.3.

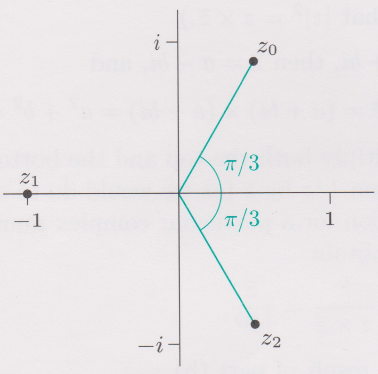


Figure S.3

Solution 5.1

From the definition of a complex exponential, we have

$$\begin{aligned} e^{-1+i\pi/3} &= e^{-1}(\cos(\pi/3) + i\sin(\pi/3)) \\ &= e^{-1}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= 0.184 + 0.319i, \end{aligned}$$

to three decimal places.

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